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THESIS

MATHEMATICAL MODELS FOR
OPERATIONAL AVAILABILITY

by

Man Won Jee

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Thesis Advisor: F. R. Richards

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Exact expressions are obtained for the availability of any series/parallel system composed of independently operating components

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Mathematical Models for Operational Availability

by

Man Won Jee
Lieutenant Colonel, Republic of Korea
B.S., Korea Military Academy, 1966
M.S., Naval Postgraduate School, 1975

Submitted in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

from the
NAVAL POSTGRADUATE SCHOOL
September 1980

ABSTRACT

The point availability of a system of components each of which is subject to random failures and has random restoration times is determined. Each component is assumed to have a fixed number of spares such that when all spares are exhausted no restoration can take place.

Exact expressions are obtained for the availability of any series/parallel system composed of independently operating components. Furthermore, expressions are obtained for the availability of series systems under five operating scenarios for which components are not independent. Order relationships on system availability for the various operating scenarios are determined.

An algorithm which optimally allocates spare parts over the components in a system is presented. The algorithm offers computational savings over solutions derived from dynamic programming.

TABLE OF CONTENTS

LIST OF SYMBOLS

λ: Constant failure rate of component parts

η: Constant replacement rate of spare parts

$F(t)$: Distribution function of lifetime of component parts $(1-e^{-\lambda t})$

$f(t)$: $\frac{dF(t)}{dt} = \lambda e^{-\lambda t}$

$G(t)$: Distribution function of replacement time of spare parts $(1-e^{-\eta t})$

$g(t)$ $\frac{dG(t)}{dt} = \eta e^{-\eta t}$

$f^{(n)}(\cdot)$: n-fold convolution of f

$f*g$: Convolution of f and g

A: Two components are in "up" states in a two-component series system when each component forms an alternating renewal process ("up" and "down")

D_1: Component #1 is in "down" state and component #2 is in "up" state in the above case (see "A")

D_2: Component #2 is in "down" state and component #1 is in "up" state in the above case (see "A")

B: Both components are in "down" states in the above case (see "A")

$r^*(s) = \int_0^\infty e^{-ts} r(t)\,dt$

π_i: Limiting probability of state i in a Markov Chain

P_i: Transition probability matrix of an embedded Markov Chain under scenario i

$q_i(j,k,s)$ = prob$\{X_n = j, X_{n+1} = k, t_{n+1} - t_n = s\}$ for scenario i where x_m is state of embedded Markov Chain after mth transition and t_m is point in time of mth transition

$[q_i^{(2n)}]_{p,q}$: Element (p,q) of 2n-fold convolution of matrix q_i

$A^{(n)}(t)$: Availability of one-component system when there are n spares available starting from state "up" at time 0

$A_{av}^{(n)}(t)$: Average availability $A_{av}^{(n)}(t) = \frac{1}{t} \int_0^t A^{(n)}(s)ds$

$B^{(n)}(t)$: Availability of one-component system when there are n spares available after time 0 starting from "down" state with original item failed on or before 0

$A_i^{(n)}(t)$: Availability of a system at time t under scenario i with every component started from "up" state at time 0 when there are n spares usable for every component of the system

$A_i^{(n,m)}(t)$: Availability of a two-component series system at time t under scenario i with both components started from "up" states at time 0 when there are n spares available for component #1 and m spares available for component #2

$A_i^{(\ell,k)}D_j(t)$: Availability of a two-component series system at time t under scenario i with the process started

from state D_j or time 0 when ℓ spares are available for component #1 and k spares are available for component #2

$$A_i(t) = \lim_{n\to\infty} A_i^{(n)}(t) = \lim_{\substack{n\to\infty \\ m\to\infty}} A_i^{(n,m)}(t)$$

$$A_i = \lim_{t\to\infty} A_i(t)$$

$A^{(\ell,m,n)}(t)$: Availability of three component system (for any configuration) at time t with every component of the system started from "up" state at time 0 when there are ℓ spares for component #1, m spares for component #2 and n spares for component #3

$A_s^{(n,m)}(t)$: Availability of two component series system at time t with both components started from "up" states when two components are mutually independent and there are n spares available for component #1 and m spares are available for component #2

$A_p^{(n,m)}(t)$: Availability of two component parallel system at time t in the above case (see $A_s^{(n,m)}(t)$)

$a_i^{(n)}(t)$: Stage i availability (stage return) in dynamic programming

$A_i(X_i)$: Maximum i-stage return in dynamic programming

$X_1 \underset{st}{\geq} X_2$: Random variable X_1 is stochastically larger than random variable X_2

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I. INTRODUCTION

A key concern of military planners is the material readiness of their weapon systems. Many attempts have been made in the past to develop measurable indicators of material readiness. Among these are reliability, fill rate, down time, maintenance manhours, supply response time, and availability. Out of these attempts to measure readiness, operational availability has become the most widely accepted indicator. Ref. [32] states, "while it is an indicator that operating forces can recognize and relate to, it also has the feature of reflecting in a logical manner the relationship of the different factors which contribute to material readiness." Much effort is underway trying to determine how weapon systems should be built, maintained, and supported with spares in order to maximize availability with given resources. This thesis examines some of the problems concerned with determining the operational availability of a weapon system and then allocating scarce resources to the system to optimize availability.

Many people talk and write about operational availability, but not all have the same notion as to what is meant by the expression. Operational availability is generally considered to be a measure of the likelihood that a system, when used under stated conditions in the actual operational environment, shall operate satisfactorily. It is clear that component reliabilities, maintenance policy, spare parts support, system

configuration, repair times, and supply response times all have impact on availability. However, this notion of operational availability can give rise to more than one workable mathematical expression. To illustrate, consider a ship on a 60-day deployment. Let A(t) be the probability that a given weapon system is operational at time t. One might use any of the following as reasonable measures of availability:

a) A(60) (end of deployment point availability)

b) $\frac{1}{60}\int_0^{60} A(t)dt$ (average availability)

c) The probability that the system shall operate satisfactorily when called upon

d) $\frac{\text{Mean time to failure (MTBF)}}{\text{MTBF + Mean time to replace (MTTR)}}$

The first expression is the likelihood that the equipment will be operational at the end of the deployment; the second represents an average availability over the duration of the deployment; the third considers only the likelihood that the system is operational at those times when needed (thus introducing the mission duty cycle as a factor to consider); and the last expression is the "definition" of availability specified by Ref. [32], and used most frequently in practice. Under certain very restrictive conditions the above expressions may yield essentially the same values. However, for most real-world cases the expressions are not equivalent; they can be substantially different. The following specific cases illustrate some differences.

Case 1: The equipment is a single component which has exponential life time with failure rate λ. There are unlimited spares aboard ship. The distribution of the sum of supply response time (time required to get the replacement part to the equipment) and repair time (hereafter called replacement time) is exponential with rate η. The system is operational at the start of the deployment.

Define the random process $\{X(t): t \geq 0\}$ as follows:

$$X(t) = \begin{cases} 1 & \text{if component is operational at } t \\ 0 & \text{otherwise} \end{cases}$$

Under the conditions above, $X(t)$ is a renewal process and $A(t) = P\{X(t) = 1]$ is easily derived from renewal theory to be

$$A(t) = \frac{\eta}{\lambda + \eta} + \frac{\lambda}{\lambda + \eta} \exp\{-(\lambda+\eta)t\} \tag{1-1}$$

or, since $\text{MTBF} = \frac{1}{\lambda}$ and $\text{MSRT} + \text{MTTR} = \frac{1}{\eta}$ we can write (1-1) as

$$A(t) = \frac{\text{MTBF}}{\text{MTBF} + (\text{MSRT} + \text{MTTR})} + \frac{(\text{MSRT} + \text{MTTR})}{\text{MTBF} + (\text{MSRT} + \text{MTTR})} \exp\{-(\lambda+\eta)t\} \tag{1-2}$$

To allow numerical comparisons of the four interpretations a), b), c), and d) of availability, let $\lambda = 0.01$ and $\eta = 0.10$. Then

a) $A(60) = 0.90921$

b) $\frac{1}{60} \int_0^{60} A(t)dt = 0.92285$

d) $\dfrac{MTBF}{MTBF + (MSRT + MTTR)} = 0.90909$

There are only minor differences between a) and d) but a bit larger difference between b) and the others. The calculation for interpretation c) yields exactly the same result as b) if the times that the component is needed are uniform on the interval (0,60]. Interpretation d) provides the most conservative estimate of availability. From equation 1-2 one can easily see that

$$\lim_{t \to \infty} A(t) = \frac{MTBF}{MTBF + (MSRT + MTTR)}$$

Thus d) is simply the limiting availability. This, however, is not always the case as is shown later.

A plot of A(t) vs. t demonstrates how point availability vaies as a function of time (length of deployment). See figure 1.1.

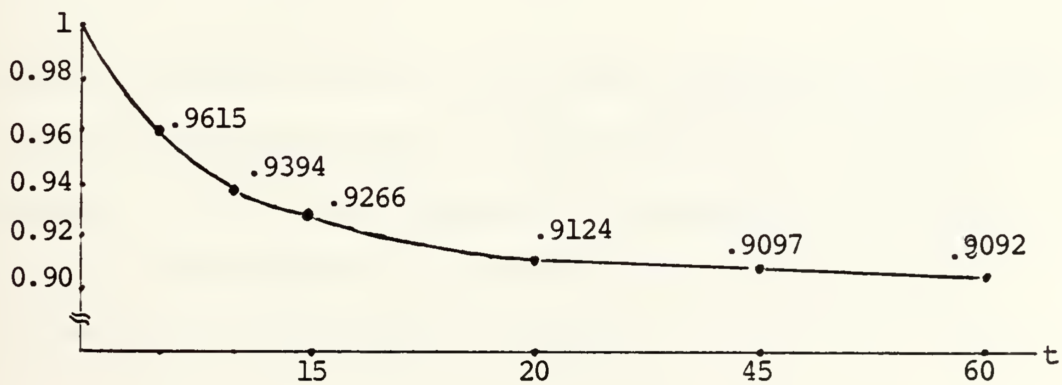


Fig. 1.1 Availability over time

Consider now a case that is more operationally realistic. Significant differences will be seen in the estimates yielded by the different expressions.

Case 2: The equipment is a single component which has exponential life time with rate λ. There is a single spare aboard ship and no chance for resupply until the deployment is over. The replacement time is exponential with rate η. The system is operational at time 0.

The random process $\{X(t), t > 0\}$ which describes the up/down status of the equipment is not now a renewal process. A sample path for $X(t)$ is shown in figure 1.2.

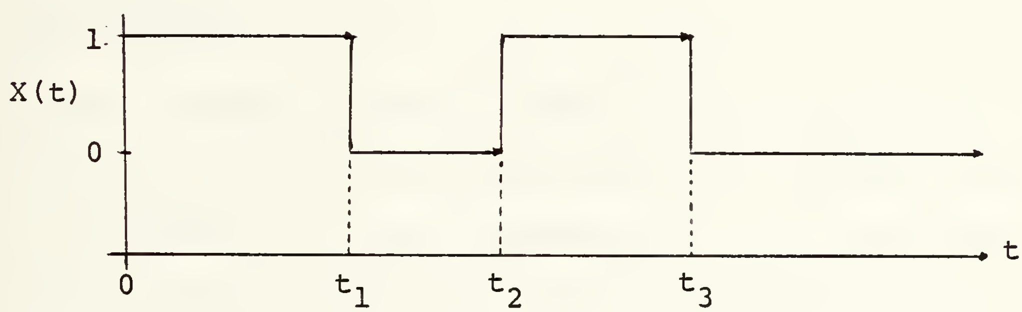


Fig. 1.2 Sample path for $X(t)$ when only one spare is stocked

The important thing to note is that the component will remain in a down status after time t_3 since there is no spare for replacement. In the next chapter an expression is derived for the point availability from which the following values are determined ($\lambda = 0.01$ and $\eta = 0.10$):

a) $A(60) = 0.8472$

b) $\frac{1}{60} \int_0^{60} A(t)dt = 0.9054$

d) $\frac{MTBF}{MTBF + MTTR} = 0.9091$

Greater differences among the availability estimates are observed when a single spare is stocked. The differences can be made quite significant by appropriate choice of the parameters λ and η. For example, if $\lambda = 0.05$ and $\eta = 0.50$,

a) $A(60) = 0.2096$

b) $\frac{1}{60} \int_0^{60} A(t)dt = 0.578361$

d) $\frac{MTBF}{MTBF + MTTR} = 0.9091$

The above examples show that there can be significant differences in the values obtained from the different definitions of availability. Each expression has its supporters. For logistics planning purposes, military planners want to determine the likelihood that a deployed unit will complete a mission with a given system operational. Therefore, they need to know the point availability at time τ where τ is the length of the deployment period. Furthermore, the average availability and the limiting availability are both functions of the point availability. Therefore, we focus primarily on point availability in this thesis. Examples of calculations of average availability and the limiting availability are provided to illustrate their calculation . The definition of

availability based on the ratio of mean time to failure to the sum of mean time to failure and mean time to replace leaves much to be desired. The definition assumes that the up and down states of the system satisfy an alternating renewal process. This implicitly assumes infinitely many spares making the expression useless for determining the number of spares required to support a system. Furthermore, the ratio expression is a limiting result, whereas, the major interest is in the availability at specified points in time or over specified intervals. Finally, the expression does not lend itself for calculating the system availability as a function of the availabilities of its components.

Let us now look at the point availabilities of some simple systems. Consider first the case in which the system is composed of two components connected in series each with infinitely many spares. We assume that the components operate independently. It is easy to see from first principles that the system availability is the product of the component availabilities. That is,

$$A_{sys}(t) = A_1(t) \cdot A_2(t)$$

Similarly, if the system is composed of two components in parallel, the system availability can easily be shown to be

$$A_{sys}(t) = 1 - (1 - A_1(t))(1 - A_2(t)) .$$

The reader familiar with reliability theory will recognize the above calculations as identical to those used to calculate reliabilities of systems composed of two components in series and parallel, respectively. These results are easily extended to any finite number of components. The formulae are also valid when each component has only finitely many spares for support. Thus, the system availability of any series/parallel/mixed system can be determined easily from the availabilities of its components provided the components operate independently.

Whenever the components of a system do not operate independently the formulae given above are not correct. There are many real world examples in which the operation of one component may depend on the operation of another component. When this happens, the calculations are much more complicated. First, the nature of the dependence must be described. Then, conditioned on the nature of the dependence, the component availabilities must be aggregated appropriately into an expression for system availability. Certainly, one cannot analyze all possible types of dependence. However, there are certain types which are most realistic from operational considerations. In this thesis, we consider explicitly four cases, called operational scenarios, in which the component operations are interrelated. The examples below demonstrate the differences in system availability caused by a change in scenario. The formulae from which the calculations are made are derived in this thesis.

Case 3:

Consider a two-component series system with both components identical. We assume that the component lifetimes and replacement times are independent and exponentially distributed with parameters $\lambda = 1/30$ and $\eta = 1/5$, respectively. We consider first the case in which the components operate independently. Let $A^{(n_1,n_2)}(t)$ be the system availability at time t when there are n_1 spares for component 1 and n_2 spares for component 2. We determine in this thesis the following availabilities for t = 90.

(n_1,n_2)	$A^{(n_1,n_2)}(90)$
(0,0)	0.0025
(1,0)	0.011
(1,1)	0.047
(2,1)	0.100
(2,2)	0.214

Case 4:

Consider now the same system configuration, component reliabilities, and component replacement times. However, now suppose that the surviving component shuts down whenever a failure occurs. Under this operational scenario, the following availabilities are obtained:

(n_1,n_2)	$A^{(n_1,n_2)}(90)$
(0,0)	0.0025
(1,0)	0.013
(1,1)	0.059
(2,1)	0.129
(2,2)	0.266

We observe that the availabilities are slightly greater than in the previous case where the components operate independently.

Case 5:

For the last case, consider the same situation described above, but assume that the identical components share a common pool of spare parts. Let $A^{(n)}(t)$ be the system availability at time t when there are n spares in the pool. The following values are determined in this thesis:

n	$A^{(n)}(t)$
0	0.0025
1	0.023
2	0.094
3	0.234
4	0.412

Given a fixed number of spares, we observe that system availability is greater when spares are shared than when each component has unique support.

The examples and discussion above illustrate that the availability of a system composed of multiple components arranged in series/parallel/mixed configurations depends on many factors. Among these are:

(1) component reliabilities

(2) component replacement times

(3) system configuration

(4) operational scenario

(5) number of spares

(6) spares sharing policy

Any effort to calculate system availability that does not explicitly consider each of these factors cannot be accurate. The expressions derived in this thesis take into account each of these factors. As a result, the expressions derived are complicated, but that is frequently the price paid for accuracy. Only after the exact expressions are known can any serious effort be made to obtain reasonable approximations and simplifications.

In the remainder of this chapter we describe the results obtained in the thesis. For mathematical simplicity and conformance to various military specifications we assume that both the lifetimes and the replacement times of the individual components are independent and identically distributed (iid) with exponential distributions.

In Chapter II, computational expressions for point availability and average availability of a one-component system having a fixed number of spares are determined.

The tradeoff of repair versus supply is examined by considering two policies; 1) a replacement policy with a fixed number, n, of spares and 2) a repair policy with repair rate η' (implicitly equivalent to an assumption of infinitely many spares). One aspect of this tradeoff is discussed by looking at what the repair rate η' must be in order to achieve the same level of availability provided by the replacement policy with n spares.

Given a fixed length of mission duration and a finite number of spares, a system may not be available at the end of a mission due to lack of spares. The probability distribution of system down time due to lack of spares is determined as a function of number of spares and mission duration.

In Chapter III, we examine two-component series systems under various operational deployment scenarios. We consider the case in which the two components operate independently nad four cases in which the components do not operate independently. The four dependent scenarios are distinguished by what happens to the surviving component when a failure occurs. The scenarios considered are all operationally realistic. We consider cases in which there are infinitely many spares and finitely many spares for each component. We also consider the case in which there are finitely many spares

which are shared by the two components. Exact results are obtained for the system point availability in some cases, and approximate results in others.

In Chapter IV, exact computational formulae are derived for the availability of series/parallel/mixed system composed of independently operating components when each component has a fixed number of spares. In addition an approximation is obtained for the availability of a two-component series system operating under the "symmetric shut down" scenario when each component has a unique finite spares pool.

In Chapter V, order relationships are established on system availability for the five operating scenarios. The exact computational expressions for the availability of a system under the non-independent scenarios are very complex. However, these order relationships can be used for establishing approximations or bounds for the exact system availability.

In Chapter VI, the results derived previously are utilized in developing an algorithm for optimally allocating a given budget for spare parts support for the components of a system. The algorithm views the system as a series/parallel mixture of independently operating modules. A module can consist of components which do not operate independently. The algorithm will handle any system for which the dependently operating components are described by the scenarios addressed in this thesis. The algorithm is developed from dynamic programming. The result, however, is an algorithm that offers substantial computational efficiency over dynamic programming.

The result is a procedure that could be utilized in the construction of allowance or load lists. A comprehensive example is presented and solved using the algorithm.

Chapter VII summarizes the results of this research effort and discusses problems of implementation. It recommends additional work for subsequent research efforts.

II. AVAILABILITY OF ONE COMPONENT SYSTEM WITH FINITE NUMBER OF SPARES

A. POINT AVAILABILITY

In this section we determine mathematical formulae for the point availability of a single component system having n spares and no repair capability. Since there are only finitely many spares and no possibility for repair, the system will alternate between up and down states until all spares are exhausted and will remain in a down state whenever the last spare fails.

Let us introduce some notation that will be used. Let T_i be the lifetime and R_i the replacement time of the ith unit. We assume that $\{T_i\}_{i=1}^{n}$ are independent and identically distributed (iid) with distribution function F(t) and probability density f(t). Similarly, the replacement times $\{R_i\}_{i=1}^{n}$ are iid with distribution G(t) and density g(t). Furthermore, the replacement times are independent of the lifetimes. We use f*g to indicate the convolution of f and g and $f^{(k)}$ to represent the k-fold convolution. Finally, let $P_k(t)$ be the probability that the kth unit will be in operation at time t.

The first result we derive is the general expression for the point availability of the single-component system when there are n spares, $A^{(n)}(t)$.

The system will be operational at time t if and only if the kth unit (k = 1, 2,...,n+1) is in operation at time t. Thus,

$$A^{(n)}(t) = \sum_{k=1}^{n+1} p_k(t) .$$

Now,

$$P_k(t) = P[\sum_{i=1}^{k-1} (T_i + R_i) \le t \text{ and } T_k + \sum_{i=1}^{k-1} (T_i + R_i) > t]$$

$$= \int_0^t (f*g)^{(k-1)}_{(s)} \bar{F}(t-s)\, ds$$

$$= [(f*g)^{(k-1)} * \bar{F}]_{(t)}$$

where

$$\bar{F}(t) = P(T_i > t].$$

We have,

$$A^{(n)}(t) = \sum_{k=1}^{n+1} [(f*g)^{(k-1)} * \bar{F})_{(t)}$$

or

$$A^{(n)}(t) = \bar{F}(t) + \sum_{k=1}^{n} [(f*g)^{(k)} * \bar{F}]_{(t)} \tag{2-1}$$

This result can be rewritten recursively as

$$A^{(n)}(t) = A^{(n-1)}(t) + [(f*g)^{(n)} * \bar{F}]_{(t)} \tag{2-2}$$

and

$$A^{(n)}(t) = \bar{F}(t) + [A^{(n-1)} * f * g]_{(t)} \tag{2-3}$$

Each expression has some usefulness. Equations (2-1) and (2-2) are probably preferable computationally. Equation (2-2) provides a simple expression for the marginal contribution that the nth spare provides to system availability.

$$\Delta^{(n)}(t) = A^{(n)}(t) - A^{(n-1)}(t) = [(f*g)^{(n)}*\bar{F}]_{(t)}$$

For the special case in which $f(t) = \lambda e^{-\lambda t}$ and $g(t) = \eta e^{-\eta t}$ the convolution expression is found using Laplace transforms to be

$$[f^{(k)}*g^{(k)}*\bar{F}]_{(t)} = [\tfrac{\theta}{\delta}]^{k}\cdot[\frac{t^{k}}{k!} + \sum_{r=1}^{k} (-1)^{r} \frac{{}_{(k+r-1)}P_{r}}{r!\ \delta^{r}}\cdot \frac{t^{(k-r)}}{(k-r)!}]e^{-\lambda t}$$

$$+ (-1)^{k+1}\cdot[\frac{\theta^{k}}{\delta^{k+1}}]\cdot[\frac{t^{(k-1)}}{(k-1)!} + \sum_{\ell=1}^{k-1} \frac{{}_{(k+\ell)}P_{\ell}}{\ell!\ \delta^{\ell}}\cdot\frac{t^{(k-\ell-1)}}{(k-\ell-1)!}]e^{-\eta t} \qquad (2-4)$$

where

$${}_{n}P_{k} = \frac{n!}{(n-k)!}, \quad \theta = \lambda\eta, \quad \text{and} \quad \delta = \eta - \lambda > 0.$$

As an example, if $\lambda = 1/30$, $\eta = 1/5$ and $t = 90$, we have

$$[(f*g)^{(7)}*\bar{F}]_{(t)} = [\tfrac{\theta}{\delta}]^{7}[\frac{t^{7}}{7!} - \frac{7}{\delta}\cdot\frac{t^{6}}{6!} + \frac{7\cdot 8}{2!\delta^{2}}\cdot\frac{t^{5}}{5!} - \frac{7\cdot 8\cdot 9}{3!\delta^{3}}\cdot\frac{t^{4}}{4!}$$

$$+ \frac{7\cdot 8\cdot 9\cdot 10}{4!\delta^{4}}\cdot\frac{t^{3}}{3!} - \frac{7\cdot 8\cdot 9\cdot 10\cdot 11}{5!\delta^{5}}\cdot\frac{t^{2}}{2!} + \frac{7\cdot 8\cdot 9\cdot 10\cdot 11\cdot 12}{6!\delta^{6}}\cdot\frac{t}{1!}$$

$$- \frac{7\cdot 8\cdots 12\cdot 13}{7!\delta^{7}}]e^{-\lambda t} + [\frac{\theta^{7}}{\delta^{8}}]\cdot[\frac{t^{6}}{6!} + \frac{8}{\delta}\cdot\frac{t^{5}}{5!} + \frac{8\cdot 9}{2!\delta^{2}}\cdot\frac{t^{4}}{4!} + \frac{8\cdot 9\cdot 10}{3!\delta^{3}}\cdot\frac{t^{3}}{3!}$$

$$+ \frac{8\cdot 9\cdot 10\cdot 11}{4!\delta^{4}}\cdot\frac{t^{2}}{2!} + \frac{8\cdot 9\cdots 11\cdot 12}{5!\delta^{5}}\cdot t + \frac{8\cdot 9\cdots 12\cdot 13}{6!\delta^{6}}]e^{-\eta t} = 0.0033$$

as the marginal contribution of the 7th spare to the availability at $t = 90$.

An observation of the convolution expression provides us with a useful approximation.

$$[(f*g)^{(n)}*\bar{F}]_{(t)} = \frac{1}{\lambda}[f^{n+1}*g^{n}]_{(t)}.$$

Now the (n+1)-fold convolution of f is the gamma density with parameters $(n+1, \frac{1}{\lambda})$, and the n-fold convolution of g is the gamma density with parameters $(n, \frac{1}{\eta})$. As n gets large these converge to normal densities with parameters $(\frac{n+1}{\lambda}, \frac{n+1}{\lambda^2})$ and $(\frac{n}{\eta}, \frac{n}{\eta^2})$, respectively. Thus, for large n, the convolution expression be be approximated by $\frac{1}{\lambda}h(t)$, where h is a normal density with mean $\frac{n+1}{\lambda} + \frac{n}{\eta}$ and variance $\frac{n+1}{\lambda^2} + \frac{n}{\eta^2}$. In terms of the standard normal density $\phi(\cdot)$ we have

$$[(f*g)^{(n)}*\bar{F}]_{(t)} \simeq \frac{\eta}{\sqrt{\eta^2(n+1)+\lambda^2 n}}\,\phi\left[\frac{t-(\frac{n+1}{\lambda}+\frac{n}{\eta})}{\sqrt{\frac{n+1}{\lambda^2}+\frac{n}{\eta^2}}}\right]$$

For the example above, with $n = 7$, $\lambda = 1/30$, $= 1/5$ and $t = 90$, we have

$$\Delta^{(7)}(90) = 0.013$$

Compared to the exact value of 0.0033 the approximation is not very good, but it would improve as n gets larger.

B. AVERAGE AVAILABILITY

Let $A_{av}^{(n)}(t)$ be the average availability, i.e.,

$$A_{av}^{(n)}(t) = \frac{1}{t}\int_0^t A^{(n)}(s)ds \ .$$

Then, from eq. (2-1), (2-2), (2-3) and the identity

$$\int \tau^m e^{a\tau} d\tau = e^{a\tau} \cdot \sum_{\rho=0}^{m} (-1)^{\rho} \cdot \frac{m!}{(m-\rho)!} \cdot \frac{\tau^{m-\rho}}{a^{\rho+1}}$$

we have

$$A_{av}^{(n)}(t) = \frac{1}{\lambda t}[1-e^{-\lambda t}] + \frac{1}{t}\sum_{k=1}^{n}\int_0^t [\frac{\theta}{\delta}]^k [\frac{\tau^k}{k!}$$

$$+ \sum_{r=1}^{k}(-1)^r \cdot \frac{(k+r-1)^{P_r}}{r!\delta^r} \cdot \frac{\tau^{(k-r)}}{(k-r)!}] e^{-\lambda\tau} d\tau$$

$$+ \frac{1}{t}\sum_{k=1}^{n}\int_0^t (-1)^{(k+1)} [\frac{\theta^k}{\delta^{k+1}}] [\frac{\tau^{k-1}}{(k-1)!}$$

$$+ \sum_{\ell=1}^{(k-1)} \frac{(k+\ell)^{P_\ell}}{\ell!\gamma^\ell} \cdot \frac{\tau^{(k-\ell-1)}}{(k-\ell-1)!}] e^{-\eta\tau} d\tau$$

$$= \frac{1}{\lambda t}[1-e^{-\lambda t}] + \frac{1}{t}\sum_{k=1}^{n}[\frac{\theta}{\delta}]^k \{ [\frac{1}{\lambda^{k+1}}$$

$$+ \sum_{\rho=0}^{k}(-1)^\rho \cdot \frac{e^{-\lambda t}}{(k-\rho)!} \cdot \frac{t^{(k-\rho)}}{(-\lambda)^{\rho+1}}] + \sum_{r=1}^{k}(-1)^r \cdot \frac{(k+r-1)^{P_r}}{r!\delta^r} \cdot$$

$$\cdot [\frac{1}{\lambda^{k-r+1}} + \sum_{\rho=0}^{k-r}(-1)^\rho \cdot \frac{e^{-\lambda t}}{(k-r-\rho)!} \cdot \frac{t^{k-r-\ell}}{(-\lambda)^{\rho+1}}]\}$$

$$+ \frac{1}{t}\sum_{k=1}^{n}(-1)^{k+1}[\frac{\theta^k}{\delta^{k+1}}]\{ [\frac{1}{\eta^k} + \sum_{\rho=0}^{(k-1)}(-1)^\rho \cdot \frac{e^{-\eta t}}{(k-1-\ell)!} \cdot$$

$$\cdot \frac{t^{(k-1-\rho)}}{(-\eta)^{\rho+1}}] + \sum_{\ell=1}^{(k-1)} \frac{(k+\ell)^{P_\ell}}{\ell!\delta^\ell}[\frac{1}{\eta^{k-\ell}}$$

$$+ \sum_{\rho=0}^{k-\ell-1}(-1)^\rho \frac{e^{-\eta t}}{(k-\ell-1-\rho)!} \cdot \frac{t^{(k-\ell-1-\rho)}}{(-\eta)^{\rho+1}}]\}. \qquad (2-5)$$

For a numerical example, let $\lambda = \frac{1}{30}$, $\eta = \frac{1}{5}$ and $t = 90$. From eqs. (2-1), (2-2), (2-3), (2-4), we determine the following values:

Table 2.1: Point Availability vs. Average Availability

	$A^{(n)}(t)$	$A_{av}^{(n)}(t)$
$\eta = 0$	0.0498	0.316737644
1	0.2170845	0.57439463
2	0.4622845	0.739965467
3	0.6721845	0.81423
4	0.791011967	0.844388
5	0.8388	0.8534416
6	0.8531	0.8555475
7	0.856412	0.856957
∞	0.859	0.8639455

C. TRADEOFF OF REPLACEMENT VS. REPAIR

A high level of operational availability can be achieved in different ways. One can increase the reliability of the system components, one can provide generous spares support, one can build in system redundancy, and one can provide a maintenance capability. After an equipment is put into operation there is little chance to do anything about component reliability or system configuration. However, one can still consider logistics tradeoffs between providing spares support and a repair capability which consumes only piece-parts support.

Suppose that a system can be repaired with repair rate η' and assume that infinitely many repairs can be made. The repair rate η' required to provide a specified level of availability, $A(t)$, can be determined by solving for η' in:

$$A_{\eta'}(t) = \frac{\eta'}{\lambda+\eta'} + \frac{\lambda}{\lambda+\eta'} \exp-(\lambda+\eta')t \qquad (2\text{-}5)$$

In evaluating the tradeoff between repair and spares support it is useful to compare the repair rate η' with the number of spares, n, required to achieve the same point availability. That is, we compare η' with the value of n which satisfies

$$A^{n}(t) = \exp(-\lambda t) + \sum_{k=1}^{n} [(f*g)^{(k)}*\bar{F}]_{(t)} \qquad (2\text{-}6)$$

Certainly, there are other factors to consider in making a decision about repair vs. replacement. For example, one would need to consider manpower requirements, training, piece-parts support needs, and space considerations. However, the tradeoff discussed above is useful as an indicator of whether or not one should even consider repair.

As a numerical example let $\lambda = \frac{1}{30}$, $\eta = \frac{1}{5}$, and $t = 90$.

$$A^{(1)}(t) = e^{-\lambda t} + (f*g*\bar{F})_{(t)} = 0.2170845$$

To achieve this level of availability we find that the maintenance rate η' should be at least 0.008233.

For n = 2,

$$A^{(2)}(t) = e^{-\lambda t} + \sum_{k=1}^{2} [(f*g)^{(k)}*\bar{F}]_{(t)} = 0.4622845 .$$

On solving (2-5) for η', with $A_{\eta'}(t)$ set to 0.4622845, we find that $\eta' = 0.028423$.

Tables 2.2 shows the comparisons between n and η' for n = 1 to 7 (for values of n larger than 7 there is very little increase in $A^{(n)}(t)$).

Table 2.2: Repair Rate as a Function of n (for a given $\{\lambda,\eta\}$)

n	A(t) Availability (λ,η)	η'	
n = 1	0.2170845	0.008233	($\eta = 24.3\eta'$)
2	0.4622845	0.028423	($\eta = 7.036\eta'$)
3	0.6721845	0.0683353	($\eta = 2.93\eta'$)
4	0.791012	0.126167	($\eta = 1.585\eta'$)
5	0.8388	0.173449	($\eta = 1.1531\eta'$)
6	0.8531	0.193578	($\eta = 1.033\eta'$)
7	0.856412	0.198812	($\eta = 1.006\eta'$)

Recall that η (the replacement rate in the finite spares case) is 0.20. The repair rate η' required to achieve the same level of system availability as achieved in the finite spares case is always less than η, but must converge to η as n gets large. The ratio between η and η' is shown in the last column of Table 2.2.

D. DISTRIBUTION OF DOWN TIME DUE TO LACK OF PARTS

In a mission of a fixed duration, the contribution of the nth spare part cannot be determined solely by looking at the point availability, This is because, even with generous spares support, the system will alternate between up and down states as the system fails and is replaced. One indicator of the contribution of the nth spare part is the decrease in downtime that results from the inclusion of the spare part. In this section we address this problem by deriving the distribution of downtime due to the lack of spare parts.

Suppose we have n-1 spares available for a component for a mission of duration [0,t]. As before let $\{T_i\}_{i=1}^{\infty}$ be i.i.d. random variables (exponential) representing "up" times and $\{R_i\}_{i=1}^{\infty}$ be i.i.d. random variables representing replacement times.

Let

$$X_1 = \sum_{i=1}^{n} T_i + \sum_{i=1}^{n-1} R_i \ .$$

X_1 is the random variable representing the duration from time 0 to the point in time at which the component fails for the nth time.

If an nth spare (n+1 parts) is added to support this component then the nth spare will begin to function after a replacement time of length R_n. Let $X_2 = X_1 + R_n$, then the actual contribution of the nth spare to the availability is made after time X_2.

Let $Y = \max\{0, t-X_2\}$. Then Y is a random variable representing the duration of "down" time due to the lack of more than n-1 spare parts. Let $k^{(n)}(t)$ be the pdf of the random variable X_2. Then

$$k^{(n)}(t) = [(f*g)^{(n)}]_{(t)}$$

and the distribution function is given by

$$L_Y(x) = P\{Y \leq x\} = P\{X_2 \geq t-x\} = \int_{t-x}^{\infty} k^{(n)}(\tau)d\tau .$$

For example, when $n = 2$,

$$k^{(2)}(\tau) = [(f*g)^{(2)}]_{(\tau)} = [\tfrac{\theta}{\delta}]^2[(\tau - \tfrac{2}{\delta})e^{-\lambda\tau} + (\tau + \tfrac{2}{\delta})e^{-\eta\tau}]$$

$$\ell^{(2)}(x) = \frac{dL(x)}{dx} = k^{(2)}(t-x)$$

$$= [\tfrac{\theta}{\delta}]^2[(t-x- \tfrac{2}{\delta})\, e^{-\lambda(t-x)} + (t-x + \tfrac{2}{\delta})e^{-\eta(t-x)}]$$

for $0 < x \leq t$, and

$$\ell^{(2)}(0) = \int_t^{\infty} k^{(2)}(\tau)d\tau = [\tfrac{\theta}{\delta}]^2[\int_t^{\infty} (\tau - \tfrac{2}{\delta})e^{-\lambda\tau}d\tau$$

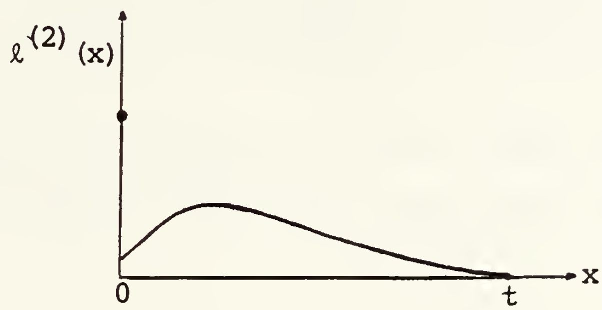
$$+ \int_t^{\infty} (\tau + \tfrac{2}{\delta})e^{-\eta\tau}d\tau]$$

since $Y = 0$ when $X_2 \geq t$.

Thus, the p.d.f. of Y when n = 2 is

$$\ell^{(2)}(x) = \begin{cases} [\frac{\theta}{\delta}]^2[(\frac{t}{\lambda} + \frac{1}{\lambda^2} - \frac{2}{\lambda\delta})e^{-\lambda t} + (\frac{t}{\eta} + \frac{1}{\eta^2} + \frac{2}{\eta\delta})e^{-\eta t}] & \text{for } X = 0 \\ \text{and} & \\ [\frac{\theta}{\delta}]^2[(t-x - \frac{2}{\delta})e^{-\lambda(t-x)} + (t-x + \frac{2}{\delta})e^{-\eta(t-x)}] & \text{for } 0 < x \le t \end{cases} \tag{2-7}$$

Graphically,



The closed form expression for the distribution function of Y for arbitrary n is:

$$L^{(n)}(x) = [\frac{\theta}{\delta}]^n \{ \frac{e^{-\lambda(t-x)}}{(n-1)!} \sum_{i=0}^{n-1} (-1)^i \frac{(n-1)!}{(n-i-1)!} \cdot \frac{(t-x)^{n-i-1}}{(-\lambda)^{i+1}}$$

$$+ \sum_{\gamma=1}^{n-1} (-1)^r \frac{(n+r-1)^P r}{r!\delta^r} \cdot \frac{e^{-\lambda(t-x)}}{(n-r-1)!} [\sum_{j=0}^{n-r-1} (-1)^j \frac{(n-r-1)!}{(n-r-j-1)!} \cdot$$

$$\cdot \frac{(t-x)^{n-r-j-1}}{(-\lambda)^{j+1}}] \} + (-1)^n [\frac{\theta}{\delta}]^n \{ \frac{e^{-\eta(t-x)}}{(n-1)!} \sum_{i=1}^{n-1} (-1)^i \cdot$$

$$\cdot \frac{(n-1)!}{(n-i-1)!} \cdot \frac{(t-x)^{n-i-1}}{(-\lambda)^{i+1}} + \sum_{r=1}^{n-1} \frac{(n+r-1)^P r}{r!\delta^r} \cdot \frac{e^{-\eta(t-x)}}{(n-r-1)!} \cdot$$

$$\cdot [\sum_{j=0}^{n-r-1} (-1)^j \cdot \frac{(n-r-1)!}{(n-r-j-1)!} \cdot \frac{(t-x)^{n-r-j-1}}{(-\lambda)^{j+1}}] \} . \qquad (2-8)$$

If we define $\overline{L}^{(n)}(x)$ to be $1 - L^{(n)}(x)$, then the expected down time in the interval $(0,t]$ when there are n spares is given by

$$\int_0^t \overline{L}^{(n)}(x)dx ,$$

and the expected amount of down time prevented by adding the nth spare part is

$$\int_0^t (\overline{L}^{(n-1)}(x) - \overline{L}^{(n)}(x))dx .$$

E. OPERATIONAL AVAILABILITY FOR A COMPONENT WITH SPARES AND A REPAIR CAPABILITY

In the previous work we have considered the case in which the single-component system had finitely many spares and no repair capability. Thus, when all the spares are used a failure will cause the system to go down and remain down until the end of the mission deployment period at which time repairs will be made or additional spares provided. Because of manpower, equipment, space, and training problems little actual repair is done during say, a deployment cruise of a ship. Thus, the finite spares/no repair problem is operationally realistic. If, however, a repair capability does exist during the mission deployment period, the number of spares required to provide a specified level of operational availability can be reduced substantially. Figure 2.1 depicts the type of situation that exists when spares and a repair capability both exist.

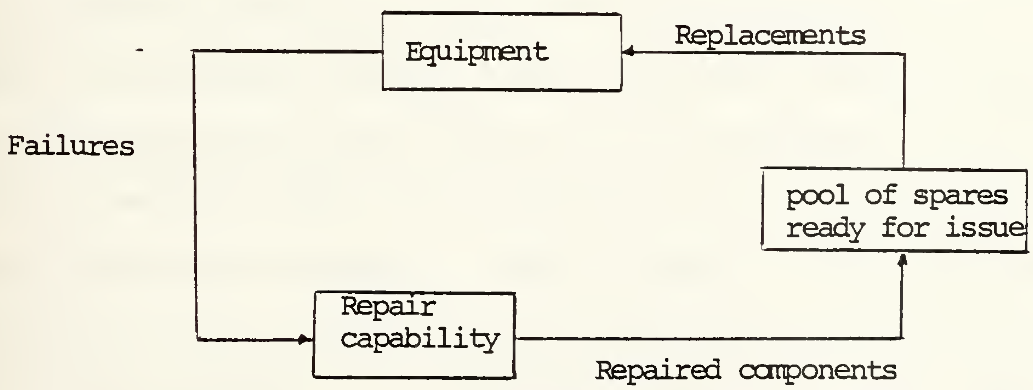


Figure 2.1: Failure/Replacement/Repair Process

Several researchers have worked on the problem in which a repair capability exists. Barlow and Proschan [5] consider the case in which the system consists of a single component with one spare, instantaneous replacement, and a single repair server. They obtain the mean system downtime, the Laplace transform of the distribution of the time to the first failure and the limiting value for system availability for general failure and repair time distributions. Gnedenko [19] and Subramanian [40] consider the case of an N component series system with a single spare and a single server repair. They obtain the Laplace transform of the availability. Subramanian and Ravichandran [42] extend these results to an arbitrary number of spares. Kumagai [23] considers a single-component system with n spares and a single server repair. He obtains an symptotic formula for stationary availability for the case of exponential lifetimes and general repair times. Gaver [17] considers the same case and obtains the Laplace transform of the probability density of the time to system failure, the mean time to system failure, and the average long-run fraction of time during which the system will be available. All of the above research assumes the replacements are instantaneous whenever spares are available. With the repair capability there are essentially infinitely many replacements possible during an interval of time. In our problem, there are only finitely many replacements possible during a given mission and the replacement times are not negligible.

A similar problem has been considered by other researchers with interest being in measures of effectiveness more closely related to supply system performance. Schrady [37], Richards [34], and Sherbrooke [39] have examined repairable item inventory models which assume that some failures will be lost to the system as unrepairable. Figure 2.2 is modified as shown in Figure 2.1 to include losses to the system and a source of external supply. The efforts of these authors have been concerned

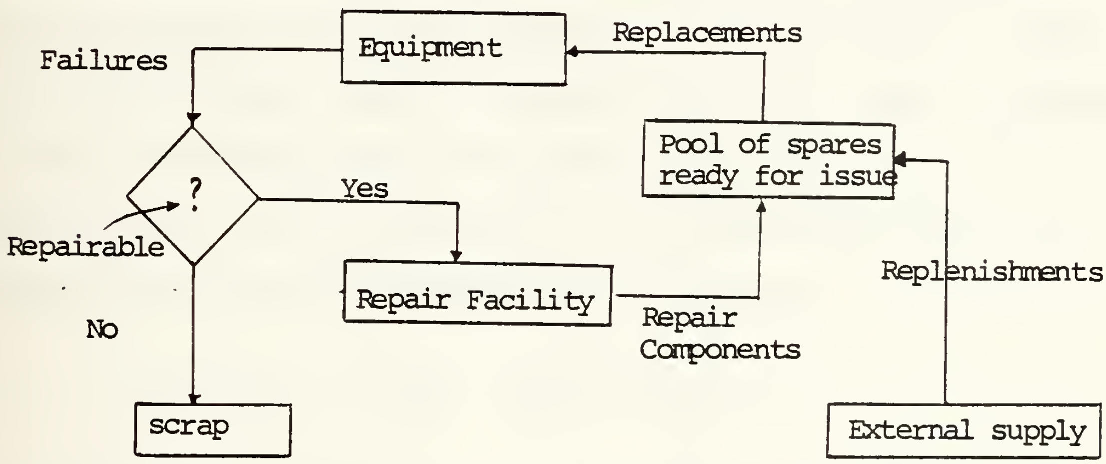


Fig. 2.2 Flow Diagram for Repairable Item System with Losses

with determining the number of spares and replenishment and repair policies so as to minimize down time during a period.

Suppose that a single-component system is supported by k spares. When the system fails, replacement takes place immediately if there is a spare available, and the failed component enters the repair process immediately. (Observe that this is equivalent to the assumption that k+1 identical units are in parallel with repair of failed units.) The system is down if and only if the number of units in the repair facility is k+1. Let

$q_i(t)$ be the probability that i units are in the repair facility at time t for $i = 1,2,\ldots,k+1$. The operational availability is then given by

$$A^{(k)}(t) = 1 - q_{k+1}(t) .$$

For the special case $k = 0$, a failure causes the system to go down and remain down until the component is repaired. The state of the system behaves as an alternating renewal process. Analysis is mathematically equivalent to our initial model with infinitely many spares and non-zero replacement times. It is well known, (see for example Ross [35]) that the operational availability for the case in which failure times are exponential iid with parameter λ and replacement times are exponential iid with parameter η is given by

$$A^{(\infty)}(t) = \frac{\eta}{\lambda+\eta} + \frac{\lambda}{\lambda+\eta} \exp(-t(\lambda+\eta))$$

On taking the limit as $t \to \infty$, the steady state operational availability is

$$\lim_{t\to\infty} A^{(\infty)}(t) = \frac{\eta}{\lambda+\eta} = \frac{\text{MTBF}}{\text{MTBF + MTTR}}$$

This is "definition" four for operational availability discussed in the Introduction. We see that, mathematically, it is correct for the case in which one is interested in the limiting result when there are iid exponential failure times, iid exponential replacement times, and infinitely many spares (or,

equivalently, 0 spares and a repair capability). For $k > 0$, the determination of the probability $q_{k+1}(t)$ is much more complicated. Choi [8], Ashar [3], Srinivasan [40], consider this problem.

In the next chapter, we extend our analysis to multi-component systems with finite spares support. We return to the situation in which no repair capability exists.

III. POINT AVAILABILITY OF TWO-COMPONENT SERIES SYSTEM WHEN TWO COMPONENTS ARE INTERDEPENDENT

A. INTRODUCTION OF SCENARIOS

A two-component series system is maintained by replacing failed components when they fail. The times to failure for type i components are exponentially distributed with parameter λ_i and the replacement times are exponentially distributed with parameter η_i. The component lifetimes are independent.

With respect to what happens to the surviving component while a failed component is being replaced the following five scenarios are defined.

Scenario 1: The surviving component continues in service, and if it fails, its replacement begins immediately and proceeds independently and concurrently with the replacement of the other failed component. All failed components resume operation as soon as they are replaced.

Scenario 2: The surviving component is shut down until replacement of the failed component is accomplished.

Scenario 3: A type-2 component continues in operation while a type-1 component is being replaced, but a type-1 component shuts down while a type-2 component is being replaced.

Scenario 4: The failure rate of the surviving component changes to $\theta_i\lambda_i$ ($\theta_i < 1$) while the failed unit is being replaced.

Scenario 5: The failure rate of the surviving component changes to $\theta_i\lambda_i$ ($\theta_i > 1$) while the failed unit is being replaced.

Very little previous work has been done for the problem in which the system components are not independent. Liang [24] considered 2-component series systems like those described as scenarios 1, 2, and 3 with infinitely many spare parts. He obtained the limiting availability for the case in which lifetimes and repair times are exponential. We are interested in determining the point availability of the systems for both the finite-spares and the infinite-spares cases.

B. SYSTEM POINT AVAILABILITY WHEN COMPONENTS ARE INDEPENDENT (SCENARIO 1)

Let $A_1(t)$ be the availability of the system composed of two components in series operating under scenario 1. We assume that both components start at time 0 from "up" states. Since the system is a series system, it will be operational at time t if and only if both components are operational. Since the components operate independently, the probability that both are operational at t is the product of the probabilities of components one and two being operational at time t. Let $a_1(t)$ and $a_2(t)$ be the component availabilities at time t, then

$$A_1(t) = a_1(t)a_2(t)$$

In fact, in the independent case, this result can be extended to a series system composed of any finite number of components. The system availability of a k-component series system (when the components operate independently) is given by

$$A_1(t) = \prod_{i=1}^{k} a_i(t)$$

This result can be extended further to include parallel systems and mixed series/parallel systems provided the components operate independently. The result for an n-component parallel system is

$$(1 - A_1(t)) = \prod_{i=1}^{n} (1 - a_i(t))$$

(The unavailabilities satisfy a product rule.) The reader, familiar with mathematical reliability will recognize these results as identical to those used to calculate the reliability of a series/parallel system configuration. In fact, the availability of a mixed system configuration is calculated just as the system reliability is determined. The component point availabilities simply replace the component reliabilities.

For the case in which components 1 and 2 in series both have infinitely many spares and exponential iid lifetimes with parameters λ_1 and λ_2, respectively and exponential iid replacement times with parameters η_1 and η_2, the system availability is:

$$A_1(t) = \left(\frac{\eta_1}{\lambda_1 + \eta_1} + \frac{\lambda_1}{\lambda_1 + \eta_1} e^{-(\lambda_1+\eta_1)t}\right) \cdot \left(\frac{\eta_2}{\lambda_2 + \eta_2} + \frac{\lambda_2}{\lambda_2 + \eta_2} e^{-(\lambda_2+\eta_2)t}\right)$$

For the case in which the components have finitely many spares, expression 2-1 from the previous chapter should be used for $a_i(t)$.

C. SYSTEM POINT AVAILABILITY FOR SCENARIO 2 (INFINITELY MANY SPARES)

Let A be the state in which both components are up;

B be the stae in which both components are down;

D_i be the state tht component i is down and the other component is up.

Let T_1 be the lifetime of the first unit of component 1 and T_2 the lifetime of the first unit of component 2. Let R_i be the replacement time of component i. Then the time to the first failure of the system is given by $\min[T_1,T_2]$ which has pdf

$$h(t) = (\lambda_1 + \lambda_2)e^{-(\lambda_1+\lambda_2)t}$$

The recurrence time of state A depends on which component fails first. If component i fails first, the recurrence time is $\min[T_1,T_2] + R_i$.

The conditional pdf of recurrence time given component i failed first is then given by

$$r_i(t) = \int_0^t (\lambda_1+\lambda_2)e^{-(\lambda_1+\lambda_2)s}\,\eta_i e^{-\eta_i(t-s)}\,ds$$

The probability that the first failure is of type 1 is

$$P[X < Y] = \int_0^\infty P[X < Y \mid X = t] f(t)\,dt$$

$$= \int_0^\infty P(Y > t) f(t)\,dt$$

$$= \int_0^\infty e^{-\lambda_2 t} \lambda_1 e^{-\lambda_1 t}\,dt$$

$$= \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

Similarly,

$$P[Y < X] = \frac{\lambda_2}{\lambda_1 + \lambda_2} .$$

Therefore, the pdf of the recurrence time of state A is given by

$$r(t) = \frac{\lambda_1}{\lambda_1 + \lambda_2} \int_0^t (\lambda_1+\lambda_2) e^{-(\lambda_1+\lambda_2)s} \eta_1 e^{-\eta_1 (t-s)}\,ds$$

$$+ \frac{\lambda_2}{\lambda_1 + \lambda_2} \int_0^t (\lambda_1 + \lambda_2) e^{-(\lambda_1+\lambda_2)s} \eta_2 e^{-\eta_2 (t-s)}\,ds$$

A set of those first recurrence times forms a renewal process. Let $h(t)$ be the renewal density function of such a process, then we have

$$A_2(t) = \bar{F}_1 \bar{F}_2 + \int_0^t h(u)\,du . \bar{F}_1(t-u) \bar{F}_2(t-u) . \qquad (3\text{-}3)$$

Taking Laplace Stieltjes transformations

$$r_1^*(s) = \frac{\lambda_1+\lambda_2}{s+\lambda_1+\lambda_2} \cdot \frac{\eta_1}{s+\eta_1}$$

$$r_2^*(s) = \frac{\lambda_1+\lambda_2}{s+\lambda_1+\lambda_2} \cdot \frac{\eta_2}{s+\eta_2}$$

$$r^*(s) = \left(\frac{\lambda_1\eta_1}{s+\eta_1} + \frac{\lambda_2\eta_2}{s+\eta_2}\right) \cdot \frac{1}{s+\lambda_1+\lambda_2}$$

$$h^*(s) = \frac{r^*(s)}{1 - r^*(s)}$$

$$h^*(s) = \frac{\lambda_1\eta_1(s+\eta_2) + \lambda_2\eta_2(s+\eta_1)}{(s+\lambda_1+\lambda_2)(s+\eta_1)(s+\eta_2)}$$

$$\cdot \frac{(s+\lambda_1+\lambda_2)(s+\eta_1)(s+\eta_2)}{(s+\lambda_1+\lambda_2)(s+\eta_1)(s+\eta_2) - \lambda_1\eta_1(s+\eta_2)\lambda_2\eta_2(s+\eta_1)}$$

$$= \frac{\lambda_1\eta_1(s+\eta_2) + \lambda_2\eta_2(s+\eta_1)}{(s+\lambda_1+\lambda_2)(s+\eta_1)(s+\eta_2)-\lambda_1\eta_1(s+\eta_2)-\lambda_2\eta_2(s+\eta_1)}$$

$$\mathcal{L}\{(h*\overline{F}_1\overline{F}_2)_{(t)}\} = h^*(s) \cdot \frac{1}{s+\lambda_1+\lambda_2}$$

$$= \frac{\lambda_1\eta_1 s + \lambda_2\eta_2 s + (\lambda_1+\lambda_2)\eta_1\eta_2}{s(s+\lambda_1+\lambda_2)[s^2+(\lambda_1+\lambda_2+\eta_1+\eta_2)s+(\eta_1\eta_2+\lambda_1\eta_2+\eta_1\lambda_2)}$$

$$= \frac{\lambda_1\eta_1 + \lambda_2\eta_2}{(s+\lambda_1+\lambda_2)(s-r_1)(s-r_2)} + \frac{(\lambda_1+\lambda_2)\eta_1\eta_2}{s(s+\lambda_1+\lambda_2)(s-r_1)(s-r_2)}$$

where

$$r_k = \frac{-(\lambda_1+\lambda_2+\eta_1+\eta_2) \pm \sqrt{(\lambda_1+\lambda_2+\eta_1+\eta_2)^2-4(\eta_1\eta_2+\lambda_1\eta_2+\eta_1\lambda_2)}}{2} \quad (k=1,2),$$

since

$$\frac{1}{(s+\lambda_1+\lambda_2)(s-r_1)(s-r_2)} = \frac{1}{(\lambda_1+\lambda_2+r_1)(\lambda_1+\lambda_2+r_2)(s+\lambda_1+\lambda_2)}$$

$$+ \frac{1}{(\lambda_1+\lambda_2+r_1)(r_1-r_2)(s-r_1)} + \frac{1}{(\lambda_1+\lambda_2+r_2)(r_2-r_1)(s-r_2)}$$

and

$$\frac{1}{s(s+\lambda_1+\lambda_2)(s-r_1)(s-r_2)} = \frac{1}{(\lambda_1+\lambda_2)r_1r_2s}$$

$$- \frac{1}{(\lambda_1+\lambda_2)(\lambda_1+\lambda_2+r_1)(\lambda_1+\lambda_2+r_2)(s+\lambda_1+\lambda_2)}$$

$$+ \frac{1}{r_1(r_1+\lambda_1+\lambda_2)(r_1-r_2)(s-r_1)} + \frac{1}{r_2(r_2+\lambda_1+\lambda_2)(r_2-r_1)(s-r_2)}$$

we have

$$\mathcal{L}^{-1}\{\frac{\lambda_1\eta_1+\lambda_2\eta_2}{(s+\lambda_1+\lambda_2)(s-r_1)(s-r_2)} + \frac{(\lambda_1+\lambda_2)\eta_1\eta_2}{s(s+\lambda_1+\lambda_2)(s-r_1)(s-r_2)}\}$$

$$= [\lambda_1\eta_1+\lambda_2\eta_2][\frac{1}{(\lambda_1+\lambda_2+r_1)(\lambda_1+\lambda_2+r_2)} e^{-(\lambda_1+\lambda_2)t}$$

$$+ \frac{1}{(r_1+\lambda_1+\lambda_2)(r_1-r_2)} e^{r_1t} + \frac{1}{(r_2+\lambda_1+\lambda_2)(r_2-r_1)} e^{r_2t}]$$

$$+ [\lambda_1+\lambda_2]\eta_1\eta_2[\frac{1}{(\lambda_1+\lambda_2)r_1r_2}$$

$$- \frac{1}{(\lambda_1+\lambda_2)(\lambda_1+\lambda_2+r_1)(\lambda_1+\lambda_2+r_2)} e^{-(\lambda_1+\lambda_2)t}$$

$$+ \frac{1}{r_1(r_1+\lambda_1+\lambda_2)(r_1-r_2)} e^{r_1t} + \frac{1}{r_2(r_2+\lambda_1+\lambda_2)(r_2-r_1)} e^{r_2t}]$$

$$\mathcal{L}^{-1}\left\{\frac{\lambda_1\eta_1 + \lambda_2\eta_2}{(s+\lambda_1+\lambda_2)(s-r_1)(s-r_2)} + \frac{(\lambda_1+\lambda_2)\eta_1\eta_2}{s(s+\lambda_1+\lambda_2)(s-r_1)(s-r_2)}\right.$$

$$= \frac{\eta_1\eta_2}{r_1r_2} + \frac{\lambda_1\eta_1+\lambda_2\eta_2-\eta_1\eta_2}{(\lambda_1+\lambda_2+r_1)(\lambda_1+\lambda_2+r_2)} e^{-(\lambda_1+\lambda_2)t}$$

$$+ \frac{(\lambda_1\eta_1 + \lambda_2\eta_2)r_1 + (\lambda_1+\lambda_2)\eta_1\eta_2}{r_1(\lambda_1+\lambda_2+r_1)(r_1-r_2)} e^{r_1t}$$

$$+ \frac{r_2(\lambda_1\eta_1 + \lambda_2\eta_2) + (\lambda_1 + \lambda_2)\eta_1\eta_2}{r_2(\lambda_1+\lambda_2+r_2)(r_2-r_1)} e^{r_2t} .$$

So

$$A_2(t) = e^{-(\lambda_1+\lambda_2)t} + \frac{\eta_1\eta_2}{r_1r_2} + \frac{\lambda_1\eta_1 + \lambda_2\eta_2 - \eta_1\eta_2}{(\lambda_1+\lambda_2+r_1)(\lambda_1+\lambda_2+r_2)} e^{-(\lambda_1+\lambda_2)t}$$

$$+ \frac{(\lambda_1\eta_1+\lambda_2\eta_2)r_1 + (\lambda_1+\lambda_2)\eta_1\eta_2}{r_1(\lambda_1+\lambda_2+r_1)(r_1-r_2)} e^{r_1t}$$

$$+ \frac{r_2(\lambda_1\eta_1+\lambda_2\eta_2) + (\lambda_1+\lambda_2)\eta_1\eta_2}{r_2(\lambda_1+\lambda_2+r_2)(r_2-r_1)} e^{r_2t} .$$

It can be shown that

$$\frac{\lambda_1\eta_1 + \lambda_2\eta_2 - \eta_1\eta_2}{(\lambda_1+\lambda_2+r_1)(\lambda_1+\lambda_2+r_2)} = -1 .$$

Finally

$$A_2(t) = \frac{\eta_1\eta_2}{r_1r_2} + \frac{(\lambda_1\eta_1 + \lambda_2\eta_2)r_1 + (\lambda_1 + \lambda_2)\eta_1\eta_2}{r_1(r_1 + \lambda_1 + \lambda_2)(r_1 - r_2)}\, e^{r_1 t}$$

$$+ \frac{(\lambda_1\eta_1 + \lambda_2\eta_2)\gamma_2 + (\lambda_1 + \lambda_2)\eta_1\eta_2}{r_2(r_2 + \lambda_1 + \lambda_2)(r_2 - r_1)}\, e^{r_2 t} \qquad (3\text{-}4)$$

D. SYSTEM POINT AVAILABILITY FOR SCENARIO 3 (INFINITELY MANY SPARES)

Let $A(t)$ be the state of the process that both components are in service at t, $B(t)$ be that both components are out of service at t, and $D_i(t)$ be that component #i is out of service with the other component in service.

Setting up a set of Kolmogorov's forward differential equations,

$$A'(t) = -(\lambda_1+\lambda_2)A(t) + \eta_1D_1(t) + \eta_2D_2(t)$$

$$D_1'(t) = \lambda_1A(t) - (\eta_1+\lambda_2)D_1(t) + \eta_2B(t)$$

$$D_2'(t) = \lambda_2A(t) - \eta_2D_2(t) + \eta_1B(t)$$

$$B'(t) = \lambda_2D_1(t) - (\eta_1+\eta_2)B(t)$$

Taking Laplace-Stieltjes transformations yields

$$sA^*(s) - A(0) = -(\lambda_1+\lambda_2)A^*(s) + \eta_1D_1^*(s) + \eta_2D_2{}^*(s)$$

$$sD_1^*(s) - D_1(0) = \lambda_1A^*(s) - (\eta_1+\eta_2)D_1^*(s) + \eta_2B^*(s)$$

$$sD_2^*(s) - D_2(0) = \lambda_2 A^*(s) - \eta_2 D_2^*(s) + \eta_1 B^*(s)$$

$$sB^*(s) - B(0) = \lambda_2 D_1^*(s) - (\eta_1+\eta_2) B^*(s)$$

using initial conditions and solving a system of linear equations for $A^*(s)$, we have

$$A^*(s) = \frac{\eta_1\eta_2}{\eta_1\eta_2+\lambda_1\eta_2+\eta_1\lambda_2 + \frac{\lambda_1\lambda_2\eta_1}{\eta_1+\eta_2+\lambda_2}} \cdot \frac{1}{s}$$

$$- \frac{\eta_1\lambda_2(\eta_2+\lambda_2-\eta_1)}{(\eta_2+\lambda_2)[\eta_1(\lambda_1+\eta_1-\lambda_2-\eta_2)-\lambda_1\lambda_2]} \cdot \frac{1}{s+r_1}$$

$$+ \frac{-r_2^3+(2\eta_1+2\eta_2+\lambda_2)r_2^2-(3\eta_1\eta_2+\eta_1^2+\eta_2^2+\eta_1\lambda_2+\eta_2\lambda_2)r_2+\eta_1\eta_2\lambda_2+\eta_1^2\eta_2+\eta_1\eta_2^2}{-r_2(r_1-r_2)(r_3-r_2)}$$

$$\cdot \frac{1}{(s+r_2)}$$

$$+ \frac{-r_3^3+(2\eta_1+2\eta_2+\lambda_2)r_3^2-(3\eta_1\eta_2+\eta_1^2+\eta_2^2+\eta_1\lambda_2+\eta_2\lambda_2)r_3+\eta_1\eta_2\lambda_2+\eta_1^2\eta_2+\eta_1\eta}{-r_3(r_1-r_3)(r_2-r_3)}$$

$$\cdot \frac{1}{(s+r_3)}$$

where

$$r_1 = (\eta_2+\lambda_2)$$

and

$$r_k = \frac{(\lambda_1+\lambda_2+2\eta_1+\eta_2)\pm\sqrt{(\lambda_1+\lambda_2+2\eta_1+\eta_2)^2-4(\eta_1\lambda_1+\eta_1\lambda_2+\eta_2\lambda_1+\eta_1\eta_2+\eta_1^2)}}{2},$$

$(k = 2,3)$, are the eigenvalues of the infinitesimal transition matrix.

Finally, the closed form solution would be

$$A_3(t) = \frac{\eta_1\eta_2}{\eta_1\eta_2+\lambda_1\eta_2+\eta_1\lambda_2+\frac{\lambda_1\lambda_2\eta_1}{\eta_1+\eta_2+\lambda_2}}$$

$$+ \frac{\eta_1\lambda_2(\eta_2+\lambda_2-\eta_1)}{(\lambda_2+\eta_2)(\eta_1\eta_2+\lambda_1\lambda_2+\eta_1\lambda_2-\eta_1\lambda_1-\eta_1^2)}\, e^{-(\lambda_2+\eta_2)t}$$

$$+ \frac{-r_2^3+(2\eta_1+2\eta_2+\lambda_2)r_2^2-(3\eta_1\eta_2+\eta_1^2+\eta_2^2+\eta_1\lambda_2+\eta_2\lambda_2)r_2+(\eta_1\eta_2\lambda_2+\eta_1^2\eta_2+\eta_1\eta_2^2)}{(-r_2)(r_1-r_2)(r_3-r_2)}$$

$$\cdot\, e^{-r_2 t}$$

$$+ \frac{-r_3^3+(2\eta_1+2\eta_2+\lambda_2)r_3^2-(3\eta_1\eta_2+\eta_1^2+\eta_2^2+\eta_1\lambda_2+\eta_2\lambda_2)r_3+(\eta_1\eta_2\lambda_2+\eta_1^2\eta_2+\eta_1\eta_2^2)}{(-r_3)(r_1-r_3)(r_2-r_3)}$$

$$\cdot\, e^{-r_3 t} \qquad (3\text{-}5)$$

Notice that all eigenvalues of the infinitesimal transition matrix

$$D = \begin{bmatrix} -(\lambda_1+\lambda_2) & \eta_1 & \eta_2 & 0 \\ \lambda_1 & -(\lambda_2+\eta_1) & 0 & \eta_2 \\ \lambda_2 & 0 & -(\lambda_1+\eta_2) & \eta_1 \\ 0 & \lambda_2 & \lambda_1 & -(\eta_1+\eta_2) \end{bmatrix},$$

r_1, r_2 and r_3 are all real. To see this, consider the square root expression found in equations for r_2 and r_3.

$$\sqrt{(\lambda_1+\lambda_2+2\eta_1+\eta_2)^2-4(\eta_1\lambda_1+\eta_1\lambda_2+\eta_2\lambda_1+\eta_1\eta_2+\eta_1^2)}$$

$$= \sqrt{(\lambda_1+\lambda_2)^2+\eta_2(\eta_2+2\lambda_2-2\lambda_1)} = \sqrt{(\lambda_1-\eta_2)^2+\lambda_2(2\lambda_1+\lambda_2+2\eta_2)}$$

which is real since all λ_i and η_i take positive values. Also it is easily seen that all eigenvalues are positive. As a result of this, the availability is known to decay exponentially with time.

E. LIMITING RESULTS FOR THE AVAILABILITIES FOR SCENARIO 1, 2, AND 3

It is simple to find limiting availabilities ($\lim_{t\to\infty} A_i(t) = A_i$) by applying results from the limiting behavior of Markov process [10]. Let π_s denote limiting probability of state s in the embedded Markov Chain.

For scenario 2, the transition probability matrix of the embedded Markov Chain having states A, D_1, D_2 is

$$P_2 = \begin{bmatrix} 0 & \frac{\lambda_1}{\lambda_1+\lambda_2} & \frac{\lambda_2}{\lambda_1+\lambda_2} \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

From the identity $\pi = \pi P_{.2}$, i.e.,

$$\pi_A = \pi_{D_1} + \pi_{D_2}$$

$$\pi_{D_1} = \frac{\lambda_1}{\lambda_1+\lambda_2} \pi_A$$

$$\pi_{D_2} = \frac{\lambda_2}{\lambda_1+\lambda_2} \pi_A ,$$

and from the limiting result of

$$A_2 = \lim_{t\to\infty} A_2(t) = \frac{\pi_A \mu_A}{\pi_A \mu_A + \pi_{D_1} \mu_{D_1} + \pi_{D_2} \mu_{D_2}} ,$$

where μ_A, μ_{D_1}, and μ_{D_2} represent the mean sojourn times of states A, D_1, and D_2, respectively. We have

$$A_2 = \frac{\pi_A \cdot \frac{1}{\lambda_1+\lambda_2}}{\pi_A \cdot \frac{1}{\lambda_1+\lambda_2} + \frac{\lambda_1}{\lambda_1+\lambda_2} \cdot \frac{1}{\eta_1} \pi_A + \frac{\lambda_2}{\lambda_1+\lambda_2} \cdot \frac{1}{\eta_2} \pi_A}$$

$$= \frac{\eta_1 \eta_2}{\eta_1 \eta_2 + \lambda_1 \eta_2 + \lambda_2 \eta_1} \qquad (3\text{-}6)$$

which we also obtain from eq. (3-4). We have used the facts that

$$\mu_A = \frac{1}{\lambda_1+\lambda_2} ,$$

$$\mu_{D_1} = \frac{1}{\eta_1}$$

and

$$\mu_{D_2} = \frac{1}{\eta_2} .$$

For scenario 3 the transition probability matrix of the embedded Markov Chain having states A, D_1, D_2, B is

$$P_3 = \begin{bmatrix} 0 & \frac{\lambda_1}{\lambda_1+\lambda_2} & \frac{\lambda_2}{\lambda_1+\lambda_2} & 0 \\ \frac{\eta_1}{\eta_1+\lambda_2} & 0 & 0 & \frac{\lambda_2}{\eta_1+\lambda_2} \\ 1 & 0 & 0 & 0 \\ 0 & \frac{\eta_1}{\eta_1+\eta_2} & \frac{\eta_2}{\eta_1+\eta_2} & 0 \end{bmatrix}$$

For the identity of $\pi = \pi P_3$

$$\pi_A = \frac{\eta_1}{\eta_1+\lambda_2} \pi_{D_1} + \pi_{D_2}$$

$$\pi_{D_1} = \frac{\lambda_1}{\lambda_1+\lambda_2} \pi_A + \frac{\eta_2}{\eta_1+\eta_2} \pi_B$$

$$\pi_{D_2} = \frac{\lambda_2}{\lambda_1+\lambda_2} \pi_A + \frac{\eta_1}{\eta_1+\eta_2} \pi_B$$

$$\pi_B = \frac{\lambda_2}{\eta_1+\lambda_2} \pi_1 .$$

Solving this system of equations in terms of π_A gives

$$\pi_{D_1} = \frac{\lambda_1(\eta_1+\eta_2)(\eta_1+\lambda_2)}{\eta_1(\lambda_1+\lambda_2)(\eta_1+\eta_2+\lambda_2)} \pi_A$$

$$\pi_{D_2} = \frac{\lambda_2(\lambda_1+\lambda_2+\eta_1+\eta_2)}{(\lambda_1+\lambda_2)(\eta_1+\eta_2+\lambda_2)} \pi_A$$

$$\pi_B = \frac{\lambda_1\lambda_2(\eta_1+\eta_2)}{\eta_1(\lambda_1+\lambda_2)(\eta_1+\eta_2+\lambda_2)} \pi_A$$

and using

$$\mu_A = \frac{1}{\lambda_1+\lambda_2},$$

$$\mu_{D_1} = \frac{1}{\eta_1+\lambda_2},$$

$$\mu_{D_2} = \frac{1}{\eta_2}$$

and

$$\mu_B = \frac{1}{\eta_1+\eta_2}$$

the limiting result

$$A_3 = \lim_{t\to\infty} A_3(t) = \frac{\pi_A\mu_A}{\pi_A\mu_A + \pi_{D_1}\mu_{D_1} + \pi_{D_2}\mu_{D_2} + \pi_B\mu_B}$$

becomes (by substitution)

$$A_3 = \frac{\eta_1\eta_2(\eta_1+\eta_2+\lambda_2)}{\eta_1\eta_2(\eta_1+\eta_2+\lambda_2)+\lambda_1\eta_2(\eta_1+\eta_2)+\eta_1\lambda_2(\lambda_1+\lambda_2+\eta_1+\eta_2)+\eta_2\lambda_1\lambda_2}$$

$$= \frac{\eta_1\eta_2}{\eta_1\eta_2 + \lambda_1\eta_2 + \eta_1\lambda_2 + \frac{\lambda_1\lambda_2\eta_1}{\eta_1+\eta_2+\lambda_2}} \qquad (3\text{-}7)$$

Note the relationship between the limiting availability expressions and "mean system up time" and "mean system down time". The limiting availability of the system is

$$A = \frac{\text{mean system up time}}{\text{mean system up time + mean system down time}} \qquad (3\text{-}8)$$

According to this formula (3-8), expressions (3-1), (3-6) and (3-7) can be written as

$$A_1 = \frac{\frac{1}{\lambda_1+\lambda_2}}{\frac{1}{\lambda_1+\lambda_2} + \frac{1}{\lambda_1+\lambda_2}[\frac{\lambda_1}{\eta_1} + \frac{\lambda_2}{\eta_2} + \frac{\lambda_1\lambda_2}{\eta_1\eta_2}]} \qquad (3\text{-}9)$$

$$A_2 = \frac{\frac{1}{\lambda_1+\lambda_2}}{\frac{1}{\lambda_1+\lambda_2} + (\frac{\lambda_1}{\eta_1} + \frac{\lambda_2}{\eta_2})\cdot\frac{1}{\lambda_1+\lambda_2}} \qquad (3\text{-}10)$$

and

$$A_3 = \frac{\frac{1}{\lambda_1+\lambda_2}}{\frac{1}{\lambda_1+\lambda_2} + \frac{1}{\lambda_1+\lambda_2}[\frac{\lambda_1(\eta_1+\eta_2)}{\eta_1(\eta_1+\eta_2+\lambda_2)} + \frac{\lambda_2(\lambda_1+\lambda_2+\eta_1+\eta_2)}{\eta_2(\eta_1+\eta_2+\lambda_2)} + \frac{\lambda_1\lambda_2}{\eta_1(\eta_1+\eta_2+\lambda_2)}]}$$

(3-11)

respectively. We can see from (3-9), (3-10) and (3-11) that mean system down time under scenario 1 is

$$\frac{1}{\lambda_1+\lambda_2}\left[\frac{\lambda_1}{\eta_1}+\frac{\lambda_2}{\eta_2}+\frac{\lambda_1\lambda_2}{\eta_1\eta_2}\right] ,$$

that of scenario 2 is

$$\frac{1}{\lambda_1+\lambda_2}\left[\frac{\lambda_1}{\eta_1}+\frac{\lambda_2}{\eta_2}\right]$$

and that of scenario 3 is

$$\frac{1}{\lambda_1+\lambda_2}\left[\frac{\lambda_1(\eta_1+\eta_2)}{\eta_1(\eta_1+\eta_2+\lambda_2)}+\frac{\lambda_2(\eta_1+\eta_2+\lambda_1+\lambda_2)}{\eta_2(\eta_1+\eta_2+\lambda_2)}+\frac{\lambda_1\lambda_2}{\eta_1(\eta_1+\eta_2+\lambda_2)}\right] .$$

Let Y_i be a random variable representing system down time under scenario i, then from the above observations we can see the relationship

$$Y_1 \overset{st}{\geq} Y_3 \overset{st}{\geq} Y_2 \qquad (3\text{-}12)$$

which is equivalent to

$$E[Y_1] \geq E[Y_3] \geq E[Y_2]$$

Also from (3-9), (3-10) and (3-11) we can see the relationship

$$A_1 \leq A_3 \leq A_2 \qquad (3\text{-}13)$$

To contrast these series system results with those of parallel system consider the limiting unavailability

$$B = \lim_{t\to\infty} B(t) = \frac{\mu_B}{\mu_{BB}} = \frac{\pi_B\mu_B}{\pi_B\mu_B+(\pi_{D_1}\mu_{D_1}+\pi_{D_2}\mu_{D_2}+\pi_A\mu_A)}$$

where μ_{BB} denotes mean recurrence time of state B.

Unavailability in this parallel system is

$$B = \frac{\text{mean system down time}}{\text{mean system down time + mean system up time}}$$

$$= \frac{\dfrac{1}{\eta_1+\eta_2}}{\dfrac{1}{\eta_1+\eta_2} + \dfrac{1}{\eta_1+\eta_2}\left[\dfrac{\eta_1}{\lambda_1} + \dfrac{\eta_2}{\lambda_2} + \dfrac{\eta_1\eta_2}{\lambda_1\lambda_2}\right]} \tag{3-14}$$

From (3-14) we can see that "mean system up time" in the two-component parallel system is

$$\frac{1}{\eta_1+\eta_2}\left[\frac{\eta_1}{\lambda_1} + \frac{\eta_2}{\lambda_2} + \frac{\eta_1\eta_2}{\lambda_1\lambda_2}\right]$$

which is very large compared with those in the series systems.

Notice that a symmetry exists in the formulae (3-9) and (3-14).

F. POINT AVAILABILITIES WHEN COMPONENTS SHARE A COMMON POOL OF SPARES

Under the assumption that the two series components share a common pool of finitely many spares, we can still use the Markov process results.

Define

$$q_i(j,k,s) \;=\; \text{probability}\{X_{n+1}=k, t_{n+1}-t_n = s/X_n = j\}$$

with states A, D_1, D_2, B for scenario i, then

$$q_1 = \begin{bmatrix} 0 & f\overline{F} & f\overline{F} & 0 \\ g\overline{F} & 0 & 0 & f\overline{G} \\ g\overline{F} & 0 & 0 & f\overline{G} \\ 0 & g\overline{G} & g\overline{G} & 0 \end{bmatrix}, \quad q_2 = \begin{bmatrix} 0 & f\overline{F} & f\overline{F} \\ q & 0 & 0 \\ q & 0 & 0 \end{bmatrix}$$

$$q_3 = \begin{bmatrix} 0 & f\overline{F} & f\overline{F} & 0 \\ g\overline{F} & 0 & 0 & f\overline{G} \\ g & 0 & 0 & 0 \\ 0 & g\overline{G} & g\overline{G} & 0 \end{bmatrix}, \quad q_k = \begin{bmatrix} 0 & f\overline{F} & f\overline{F} & 0 \\ g\hat{\overline{F}} & 0 & 0 & \hat{f}\overline{G} \\ g\hat{\overline{F}} & 0 & 0 & \hat{f}\overline{G} \\ 0 & g\overline{G} & g\overline{G} & 0 \end{bmatrix},$$

$$k = 4,5$$

where $\hat{f}(t) = \theta\lambda e^{-\theta\lambda t}$. In scenario 2, state B does not exist.

Each element of the above matrices represents the p.d.f. of sojourn time of state j multiplied by the probability that the next state the process will visit is k. For example, in q_3 the element $f\overline{G}(s)$ $(= \lambda e^{-(\lambda+\eta)s})$ could be written as $(\eta+\lambda)e^{-(\eta+\lambda)s} \cdot \frac{\lambda}{\eta+\lambda}$. This means that at time s the process leaves state D_1 and the probability that state B will occur before state A occurs is $\frac{\lambda}{\lambda+\eta}$.

If the system is available at t with the kth spare alive at t then 2k transitions have been made before t since each spare used

requires 2 transitions. Hence availability of the system under scenario i with n common spares is

$$A_i^{(n)}(t) = A_i^{(n-1)}(t) + \int_0^t [q_i^{(2n)}]_{1.1(s)} \cdot \bar{F}^2(t-s)ds$$

$$= A_i^{(n-1)}(t) + \{[q_i^{(2n)}]_{1.1} * \bar{F}^2\}_{(t)} \qquad (3\text{-}15)$$

where $[q_i^{(2n)}]_{1.1}$ is (1.1) element of 2n-fold convolution of the matrix q_i. This recursive formula can also be written as

$$A_i^{(n)}(t) = \bar{F}^2(t) + \sum_{j=1}^{n} \int_0^t [q_i^{(2j)}]_{1.1(s)} \cdot \bar{F}^2(t-s)ds$$

$$= \bar{F}^2(t) + \sum_{j=1}^{n} \{[q_i^{(2j)}]_{1.1} * \bar{F}^2\}_{(t)} \qquad (3\text{-}16)$$

Notice that $\{[q_i^{(2j)}]_{1.1} * \bar{F}^2\}_{(t)}$ represents the marginal contribution to the availability of the jth spare. For example, if n = 2

$$A_1^{(2)}(t) = \bar{F}^2(t) + 2[f\bar{F} * g\bar{F} * \bar{F}^2]_{(t)} + 4[(f\bar{F})^{(2)} * (g\bar{F})^{(2)} * \bar{F}^2]_{(t)}$$

$$+ 4[f\bar{F} * f\bar{G} * g\bar{F} * g\bar{G} * \bar{F}^2]_{(t)}$$

$$A_2^{(2)}(t) = \bar{F}^2(t) + 2[f\bar{F} * g * \bar{F}^2]_{(t)} + 4[(f\bar{F})^{(2)} * g^{(2)} * \bar{F}^2]_{(t)}$$

$$A_3^{(2)}(t) = \bar{F}^2(t) + [f\bar{F}*g\bar{F}*\bar{F}^2]_{(t)} + [f\bar{F}*g*\bar{F}^2]_{(t)}$$

$$+ [(f\bar{F})^{(2)}*g^{(2)}*\bar{F}^2]_{(t)} + 2[(f\bar{F})^{(2)}*g*g\bar{F}*\bar{F}^2]_{(t)}$$

$$+ [(f\bar{F})^{(2)}*(g\bar{F})^{(2)}*\bar{F}^2]_{(t)} + [f\bar{F}*f\bar{G}*g\bar{G}*g\bar{F}*\bar{F}^2]_{(t)}$$

$$+ [f\bar{F}*f\bar{G}*g\bar{G}*g*\bar{F}^2]_{(t)}$$

or alternatively

$$A_1^{(2)}(t) = A_1^{(1)}(t) + 4[(f\bar{F})^{(2)}*(g\bar{F})^{(2)}*\bar{F}^2]_{(t)}$$

$$+ 4[f\bar{F}*f\bar{G}*g\bar{F}*g\bar{G}*\bar{F}^2]_{(t)}$$

$$A_2^{(2)}(t) = A_2^{(1)}(t) + 4[(f\bar{F})^{(2)}*g^{(2)}*\bar{F}^2]_{(t)}$$

$$A_3^{(3)}(t) = A_3^{(1)}(t) + [(f\bar{F})^{(2)}*g^{(2)}*\bar{F}^2]_{(t)}$$

$$+ [(f\bar{F})^{(2)}*g*g\bar{F}*\bar{F}^2]_{(t)} + [(f\bar{F}^{(2)}*(g\bar{F}^{(2)}*\bar{F}^{(2)}]_{(t)}$$

$$+ [f\bar{F}*f\bar{G}*g\bar{G}*g\bar{F}*\bar{F}^2]_{(t)} + [f\bar{F}*f\bar{G}*g\bar{G}*g*\bar{F}^2]_{(t)}$$

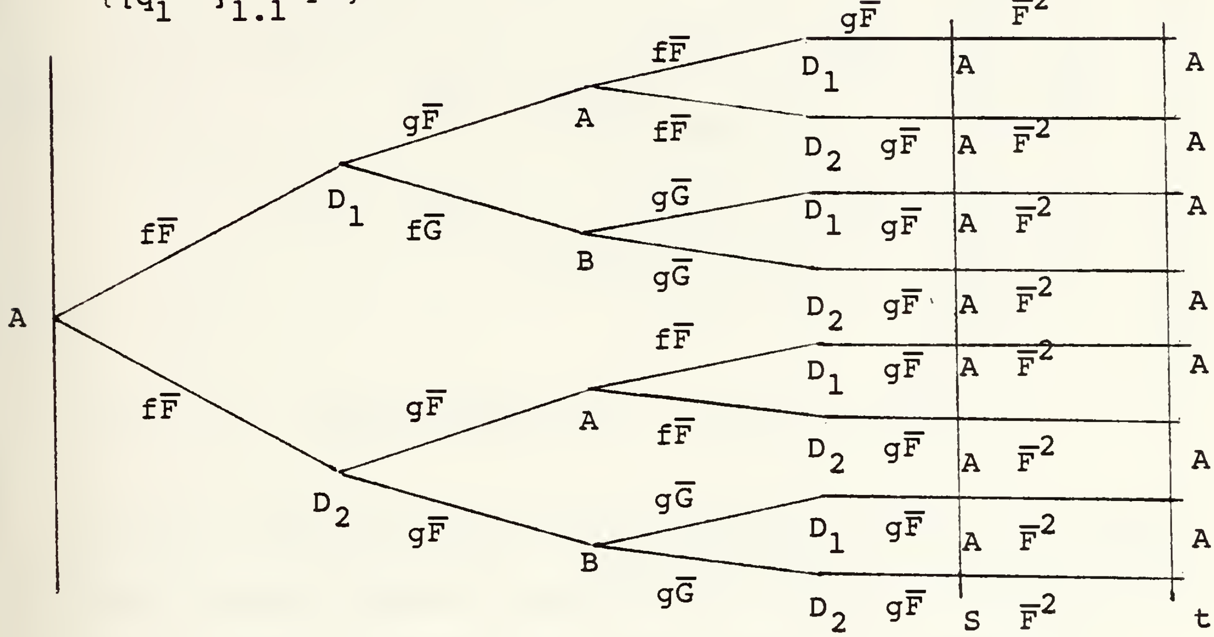
Here

$$[A_i^{(2)}(t) - A_i^{(1)}(t)] = \{[q_i^{(4)}]_{1.1}*\bar{F}^2\}_{(t)}$$

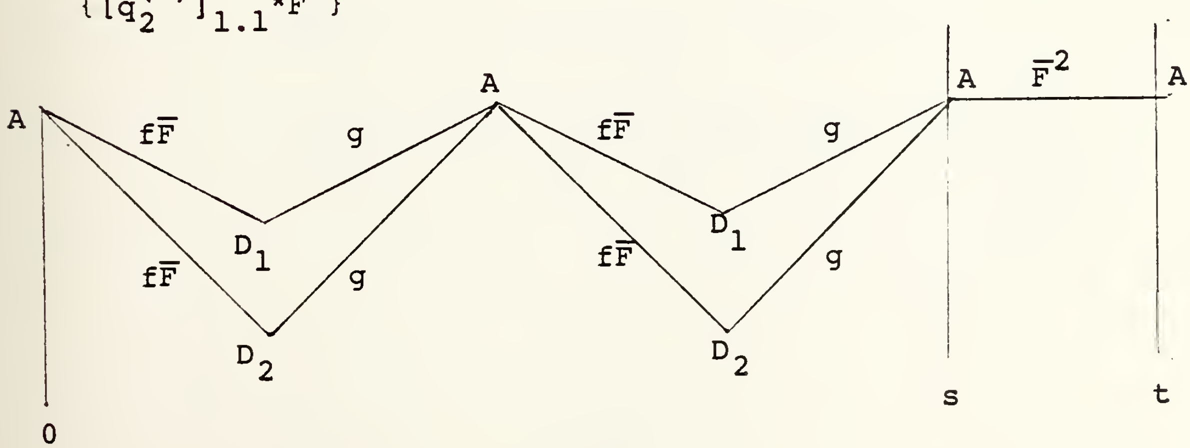
is the marginal contribution to the availability of the second

spare. The sequences of states that the sample paths under scenario i will follow for the term $\{[q_i^{(4)}]_{1.1} * \bar{F}^2\}_{(t)}$ can be illustrated pictorially as follows.

$\{[q_1^{(4)}]_{1.1} * \bar{F}^2\}$



$\{[q_2^{(4)}]_{1.1} * \bar{F}^2\}$



$\{[q_3^{(4)}]_{1.1} * \bar{F}^2\}$

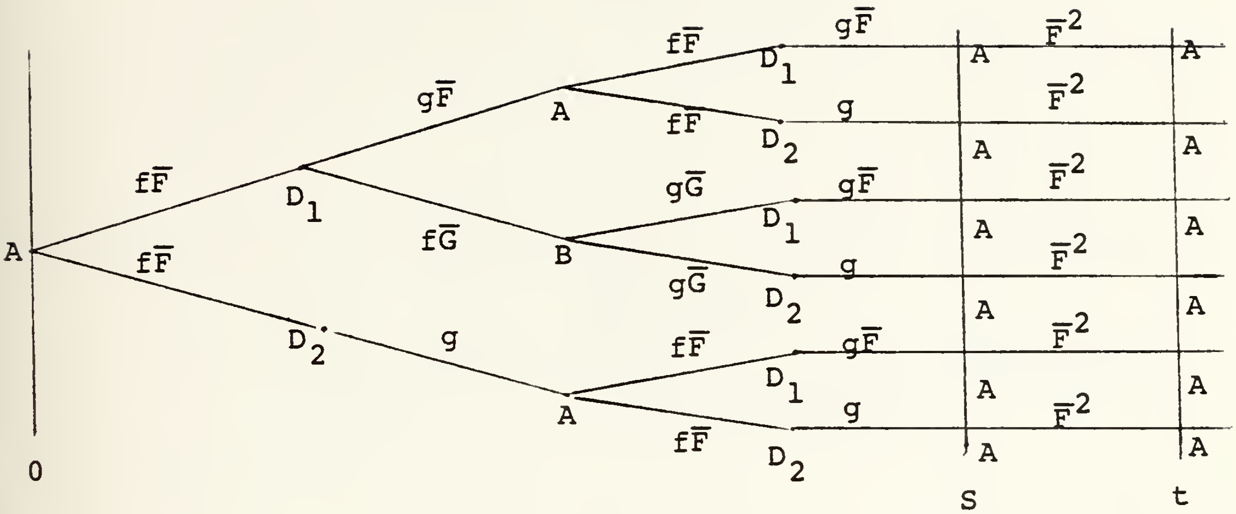
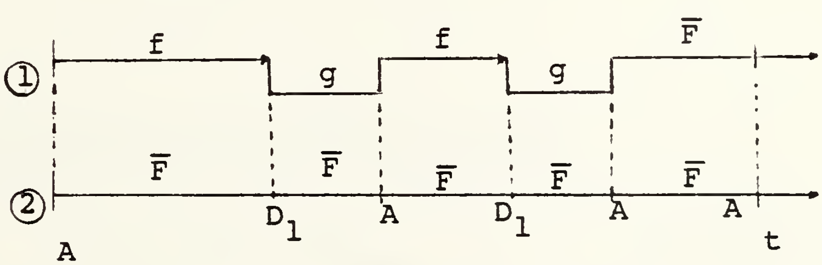


Fig. 3.1 Sample paths for $\{[q_i^{(4)}]_{1.1} * \bar{F}^2\}_{(t)}$

Closer inspection of the term $\{[q_1^{(4)}]_{1.1} * \bar{F}^2\}_{(t)}$ is useful. In order for the system to be alive at t with the second spare operational the sequences of states the sample paths for this term will follow can be categorized as following eight cases. Since a symmetry exists by the change of role between component #1 and #2, the illustration of four cases is enough

Case I.



Case II

Case III

Case IV

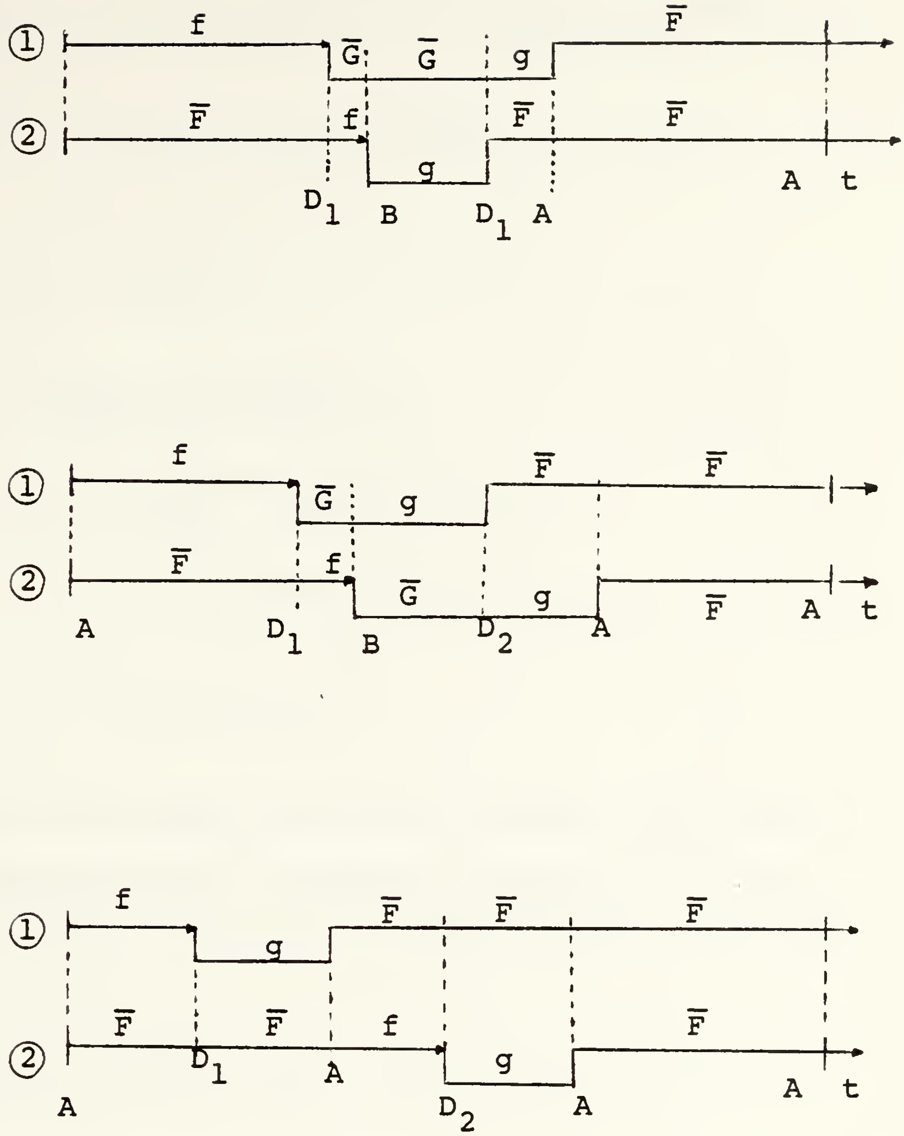


Fig. 3.2 Patterns of sample path for $\{[q_1^{(4)}]_{1.1} * \bar{F}^2\}_{(t)}$

Computing the probability of each case yields

$$\text{Prob}\{\text{Case I}\} = \bar{F}(t) \cdot [f^{(2)} * g^{(2)} * \bar{F}]_{(t)} = [(f\bar{F}^{(2)} * (g\bar{F})^{(2)} * \bar{F}^{(2)}]_{(t)}$$

$$\text{Prob}\{\text{Case II}\} = \text{Prob}\{\text{Case III}\} = [f\bar{F}*f\bar{G}*g\bar{F}*g\bar{G}*\bar{F}^2]_{(t)}$$

$$\text{Prob}\{\text{Case IV}\} = [(f\bar{F})^{(2)}*(g\bar{F})^{(2)}*\bar{F}^2]_{(t)}$$

Notice that by the summetry

$$2\cdot\text{Prob}\{\text{Case II} \cup \text{Case III} \cup \text{Case IV}\}$$

$$= 2[(f\bar{F})^{(2)}*(g\bar{F})^{(2)}*\bar{F}^2]_{(t)}$$

$$+ 4[f\bar{F}*f\bar{G}*g\bar{F}*g\bar{G}*\bar{F}^2]_{(t)}$$

$$= [(f*g*\bar{F})_{(t)}]^2$$

which represents the probability that starting from state A at time 0 the process will be found in state A at time t with exactly one spare replaced in each component some time before t.

To see the last equivalence, recall from (2-3) that

$$\{[f*g*\bar{F}]_{(t)}\}^2 = [\frac{\theta}{\delta^2} e^{-\eta t} + \frac{\theta}{\delta} te^{-\lambda t} - \frac{\theta}{\delta^2} e^{-\lambda t}]^2 .$$

Now computing the other terms gives

$$[(f\bar{F})^{(2)}*(g\bar{F})^{(2)}*\bar{F}^2]_{(t)} = \mathcal{L}^{-1}[\frac{(\eta\lambda)^2}{(s+2\lambda)^3(s+\eta+\lambda)}] .$$

$$\frac{(\eta\lambda)^2}{(s+2\lambda)^3(s+\eta+\lambda)^2} = \frac{A}{(s+2\lambda)} + \frac{B}{(s+2\lambda)^2} + \frac{C}{(s+2\lambda)^3} + \frac{D}{(s+\eta+\lambda)} + \frac{E}{(s+\eta+\lambda)^2}$$

where

$$A = \frac{3\theta^2}{\delta^4}, \quad B = -\frac{2\theta^2}{\delta^3}, \quad C = \frac{\theta^2}{\delta^2}, \quad D = -\frac{3\theta^2}{\delta^4},$$

$$E = -\frac{\theta^2}{\delta^3}.$$

Inverting the Laplace transforms

$$[(f\bar{F})^{(2)} * (g\bar{F})^{(2)} * \bar{F}^2]_{(t)} = \frac{3\theta^2}{\delta^4} e^{-2\lambda t} - \frac{2\theta^2}{\delta^3} te^{-2\lambda t} + \frac{\theta^2}{\delta^2} \cdot \frac{t^2}{2} e^{-2\lambda t}$$

$$- \frac{3\theta^2}{\delta^4} e^{-(\eta+\lambda)t} - \frac{\theta^2}{\delta^3} te^{-(\eta+\lambda)t}.$$

Similarly

$$[f\bar{F}*f\bar{G}*g\bar{F}*g\bar{G}*\bar{F}^2]_{(t)} = \mathcal{L}^{-1}\left[\frac{(\eta\lambda)^2}{(s+2\lambda)^2(s+2\eta)(s+\eta+\lambda)^2}\right]$$

$$= -\frac{5\theta^2}{4\delta^4} e^{-2\lambda t} + \frac{\theta^2}{2\delta^3} te^{-2\lambda t} + \frac{\theta^2}{4\delta^4} e^{-2\eta t}$$

$$+ \frac{\theta^2}{\delta^4} e^{-(\eta+\lambda)t} + \frac{\theta^2}{\delta^3} te^{-(\eta+\lambda)t}$$

By substitution we have the above equivalence.

G. EXTENSIONS OF SCENARIOS FOR MORE THAN TWO-COMPONENT SERIES SYSTEMS WHEN SPARES ARE SHARED

Since all the components in this system share the same spare parts we can formulate a semi-Markov kernel [9] over the state space according to the given scenario.

Taking an example of a three-component system will illustrate the extension. Let D_{ijk} be the state that component i, j and k are not in service, and consider the following extensions of scenarios.

Case 1: Extension of scenario 2, i.e., if any of the three components fails then the others will be shut down. We have a semi-Markov-kernel q_2 having states A, D_1, D_2 and D_3.

$$q_2 = \begin{pmatrix} 0 & f\bar{F}^2 & f\bar{F}^2 & f\bar{F}^2 \\ g & 0 & 0 & 0 \\ g & 0 & 0 & 0 \\ g & 0 & 0 & 0 \end{pmatrix}$$

Case 2: Extensions of scenario 3.

Consider the following four subcases:

Case 2-a: If component #1 fails first the others will be shut down, but the operations of component #1 are not affected by failures of others, then the semi-Markov kernel q_{2a} will be

$$q_{2a} = \begin{pmatrix} 0 & f\bar{F}^2 & f\bar{F}^2 & f\bar{F}^2 & 0 & 0 & 0 & 0 \\ g & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ g\bar{F}^2 & 0 & 0 & 0 & f\overline{FG} & 0 & f\overline{FG} & 0 \\ g\bar{F}^2 & 0 & 0 & 0 & 0 & f\overline{FG} & f\overline{FG} & 0 \\ 0 & g\overline{FG} & g\overline{FG} & 0 & 0 & 0 & 0 & f\bar{G}^2 \\ 0 & g\overline{FG} & 0 & g\overline{FG} & 0 & 0 & 0 & f\bar{G}^2 \\ 0 & 0 & g\overline{FG} & g\overline{FG} & 0 & 0 & 0 & f\bar{G}^2 \\ 0 & 0 & 0 & 0 & g\bar{G}^2 & g\bar{G}^2 & g\bar{G}^2 & 0 \end{pmatrix}$$

over the states A, D_1, D_2, D_3, D_{12}, D_{13}, D_{23} and D_{123}.

Case 2-b: If component #1 or component #2 fails, component #3 will be shut down, but a failure of component #3 does not affect the operations of component #1 or component #2. In this case the semi-Markov kernel is

$$q_{3b} = \begin{pmatrix} 0 & f\bar{F}^2 & f\bar{F}^2 & f\bar{F}^2 & 0 & 0 & 0 & 0 \\ g\bar{F} & 0 & 0 & 0 & f\bar{G} & 0 & 0 & 0 \\ g\bar{F} & 0 & 0 & 0 & f\bar{G} & 0 & 0 & 0 \\ g\bar{F}^2 & 0 & 0 & 0 & 0 & f\bar{F}\bar{G} & f\bar{F}\bar{G} & 0 \\ 0 & g\bar{F}\bar{G} & 0 & g\bar{F}\bar{G} & 0 & 0 & 0 & f\bar{G}^2 \\ 0 & 0 & g\bar{F}\bar{G} & g\bar{F}\bar{G} & 0 & 0 & 0 & f\bar{G}^2 \\ 0 & 0 & 0 & 0 & g\bar{G}^2 & g\bar{G}^2 & g\bar{G}^2 & 0 \end{pmatrix}$$

over the states A, D_1, D_2, D_3, D_{12}, D_{13}, D_{23} and D_{123}.

Case 3-c: The component #3 shuts down only when both of components #1, and #2 are not in service. The semi-Markov kernel in this case is

$$q_{3c} = \begin{pmatrix} 0 & f\bar{F}^2 & f\bar{F}^2 & f\bar{F}^2 & 0 & 0 & 0 & 0 \\ g\bar{F}^2 & 0 & 0 & 0 & f\bar{F}\bar{G} & f\bar{F}\bar{G} & 0 & 0 \\ g\bar{F}^2 & 0 & 0 & 0 & f\bar{F}\bar{G} & 0 & f\bar{F}\bar{G} & 0 \\ g\bar{F}^2 & 0 & 0 & 0 & 0 & F\bar{F}\bar{G} & f\bar{F}\bar{G} & 0 \\ 0 & g\bar{G} & g\bar{G} & 0 & 0 & 0 & 0 & 0 \\ 0 & g\bar{F}\bar{G} & 0 & g\bar{F}\bar{G} & 0 & 0 & 0 & f\bar{G}^2 \\ 0 & 0 & g\bar{F}\bar{G} & g\bar{F}\bar{G} & 0 & 0 & 0 & f\bar{G}^2 \\ 0 & 0 & 0 & 0 & g\bar{G}^2 & g\bar{G}^2 & g\bar{G}^2 & 0 \end{pmatrix}$$

over the same states as in case 3-b.

Case 3-d: If any two of the components fail the other component shuts down. The semi-Markov kernel would then be

$$q_{3d} = \begin{bmatrix} 0 & f\bar{F}^2 & f\bar{F}^2 & f\bar{F}^2 & 0 & 0 & 0 & 0 \\ g\bar{F}^2 & 0 & 0 & 0 & f\overline{FG} & f\overline{FG} & 0 & 0 \\ g\bar{F}^2 & 0 & 0 & 0 & f\overline{FG} & 0 & f\overline{FG} & 0 \\ g\bar{F}^2 & 0 & 0 & 0 & 0 & f\overline{FG} & f\overline{FG} & 0 \\ 0 & g\bar{G} & g\bar{G} & 0 & 0 & 0 & 0 & 0 \\ 0 & g\bar{G} & 0 & g\bar{G} & 0 & 0 & 0 & 0 \\ 0 & 0 & g\bar{G} & g\bar{G} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & g\bar{G} & g\bar{G} & g\bar{G} & 0 \end{bmatrix}$$

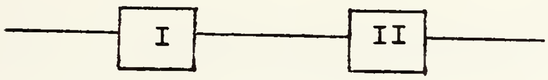
over the same states as in case 3-b.

Once we set up a semi-Markov kernel, $A_i^{(n)}(t)$ is obtained by eq. (3-16). Actual evaluations get harder as n increases in eq. (3-16). In cases such as these approximation formulae or availability bounds would be necessary.

IV. COMPUTATIONAL FORMULAE FOR THE AVAILABILITIES OF DIFFERENT SYSTEM CONFIGURATIONS

A. INDEPENDENT TWO-COMPONENT SERIES SYSTEM

1. Case of No Sharing of Spares



$\{\lambda_1, \eta_1\}$ $\{\lambda_2, \eta_2\}$

Denote by $A_s^{(n,m)}(t)$ the availability of the system when component #1 has n spares and component #2 has m spares. Both components start their operations from "up" states.

Denote by $A_c^{(j)}(t)$ (c = I,II) the availability of component c when j spares are available. Then

$$A_s^{(n,m)}(t) = A_I^{(n)}(t) \cdot A_{II}^{(m)}(t) \tag{4-1}$$

Suppose we add one more spare to component I (n+1th spare). The marginal increase in the availability of component I would be $[(f_1 * g_1)^{(n+1)} * \bar{F}_1]_{(t)}$. Hence

$$\begin{aligned} A_s^{(n+1,m)}(t) &= \{A_I^{(n)}(t) + [(f_1 * g_1)^{(n+1)} * \bar{F}_1]_{(t)}\} \cdot A_{II}^{(m)}(t) \\ &= A_s^{(n,m)}(t) + [(f_1 * g_1)^{(n+1)} * \bar{F}_1]_{(t)} \cdot A_{II}^{(m)}(t) \end{aligned} \tag{4-2}$$

Also,

$$A_s^{(n,m+1)}(t) = A_s^{(n,m)}(t) + [(f_2*g_2)^{(m+1)}*\bar{F}^2]_{(t)} \cdot A_I^{(n)}(t) \tag{4-3}$$

When we add (1.1) spares to (n,m),

$$A_s^{(n+1,m+1)}(t) = A_s^{(n,m)}(t) + [(f_1*g_1)^{(n+1)}*\bar{F}_1]_{(t)} \cdot A_{II}^{(m)}(t)$$

$$+ [(f_2*g_2)^{(m+1)}*\bar{F}_2]_{(t)} \cdot A_I^{(n)}(t)$$

$$+ [(f_1*g_1)^{(n+1)}*\bar{F}_1]_{(t)} \cdot [(f_2*g_2)^{(m+1)}*\bar{F}_2]_{(t)} \tag{4-4}$$

The second term of the R.H.S. of eq. (4-4) is the contribution made by the (n+1) st spare of component I to the system availability, the third term is the contribution of component II; and the last term is made jointly by the (n+1)st and (m+1) st spares.

2. <u>Case of Sharing of Spares</u>

$\{\lambda,\eta\}$ $\{\lambda,\eta\}$

Assume there are n spares available for both components. Hence, if component I uses k ($0 \leq k \leq n$) spares during $[0,t]$ then component II can use up to (n-k) spares. Let $A^{\{ij\}}(t)$ be the probability that the system is available at time t with (i,j)th spares alive at t. Then

$$A_s^{(0)}(t) = A_s^{\{0,0\}}(t) = e^{-2\lambda t}$$

$$A_s^{(1)}(t) = A_s^{\{0,0\}}(t) + A_s^{\{1,0\}}(t) + A_s^{\{0,1\}}$$

$$= A_s^{(0)}(t) + 2[f*g*\bar{F}]_{(t)} \cdot \bar{F}(t)$$

$$A_s^{(2)}(t) = A_s^{\{0,0\}}(t) + A_s^{\{1,0\}}(t) + A_s^{\{0,1\}}(t) + A_s^{\{0,2\}}(t)$$

$$+ A_s^{\{1.1\}}(t) + A_s^{\{2,0\}}(t) \quad .$$

$$= A_s^{(1)}(t) + 2[(f*g)^{(2)}*\bar{F}]_{(t)} \cdot \bar{F}(t) + \{[f*g*\bar{F}]_{(t)}\}^2 \; .$$

One schematic expression for the possible events which comprise $A_s^{(n)}(t)$ is shown below for $n = 4$ $(n+1 = 5)$.

Component I:	0,	1,	2,	3,	4,	5*
Component II:	0	0	0	0	0	0*
	1	1	1	1	1*	
	2	2	2	2*		
	3	3	3*			
	4	4*				
	5*					

Therefore,

$$A_s^{(5)}(t) = A_s^{(4)}(t) + A_s^{\{0.5\}}(t) + A_s^{\{1,4\}}(t) + A_s^{\{2,3\}}(t)$$

$$+ A_s^{\{3,2\}}(t) + A_s^{\{4,1\}}(t) + A_s^{\{5,0\}}(t)$$

$$A_s^{(5)}(t) = A_s^{(4)}(t) + 2\bar{F}(t)\cdot[(f*g)^{(5)}*\bar{F}]_{(t)}$$

$$+ 2[f*g*\bar{F})_{(t)}\cdot[(f*g)^{(4)}*\bar{F}]_{(t)}$$

$$+ 2[(f*g)^{(2)}*\bar{F}]_{(t)}\cdot[(f*g)^{(3)}*\bar{F}]_{(t)}$$

$$= A_s^{(4)}(t) + \sum_{i=0}^{5}[(f*g)^{(i)}*\bar{F}]_{(t)}\cdot[(f*g)^{(5-i)}*\bar{F}]_{(t)}$$

where

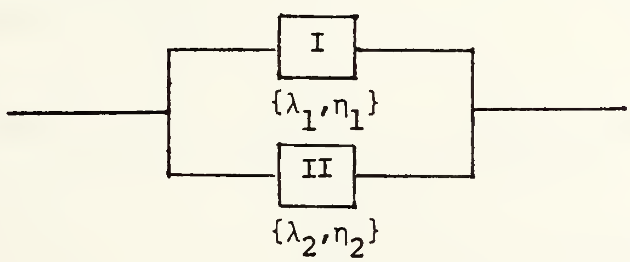
$$[(f*g)^{(0)}*\bar{F}]_{(t)} = \bar{F}(t) .$$

In general,

$$A_s^{(n+1)}(t) = A_s^{(n)}(t) + \sum_{i=0}^{(n+1)} [(f*g)^{(i)}*\bar{F}]_{(t)} \cdot [(f*g)^{(n+1-i)}*\bar{F}]_{(t)} \qquad (4\text{-}5)$$

B. INDEPENDENT TWO-COMPONENT PARALLEL SYSTEM

1. <u>Case of No Sharing of Spares</u>



Denote by $A_p^{(n,m)}(t)$ the availability of this parallel system when (n,m) spares are available. Then

$$A_p^{(n,m)}(t) = A_I^{(n)}(t) + A_{II}^{(m)}(t) - A_I^{(n)}(t)\cdot A_{II}^{(m)}(t) \qquad (4\text{-}6)$$

If we add one more spare to component I, then

$$\begin{aligned} A_p^{(n+1,m)}(t) = {} & \{A_I^{(n)}(t) + [(f_1 * g_1)^{(n+1)} * \bar{F}]_{(t)}\} \\ & + A_{II}^{(m)}(t) - \{A_I^{(n)}(t) \\ & + [(f_1 * g_1)^{(n+1)} * \bar{F}_1]_{(t)}\}\cdot A_{II}^{(m)}(t) \end{aligned}$$

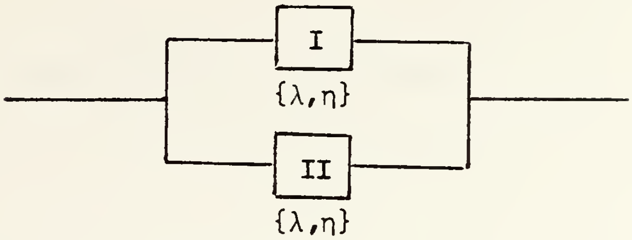
$$\begin{aligned} A_p^{(n+1,m)}(t) = {} & A_p^{(n,m)}(t) + [(f_1 * g_1)^{(n+1)} * \bar{F}_1]_{(t)} \\ & - [(f_1 * g_1)^{(n+1)} * \bar{F}_1]_{(t)}\cdot A_{II}^{(m)}(t) \qquad (4\text{-}7) \end{aligned}$$

$$\begin{aligned} A_p^{(n,m+1)}(t) = {} & A_p^{(n,m)}(t) + [(f_2 * g_2)^{(m+1)} * \bar{F}_2]_{(t)} \\ & - [(f_2 * g_2)^{(m+1)} * \bar{F}_2]_{(t)}\cdot A_I^{(n)}(t) . \qquad (4\text{-}8) \end{aligned}$$

The marginal contribution made by the (n+1)th spare of component I is $A_p^{(n+1,m)}(t) - A_p^{(n,m)}(t)$, and that of the (m+1)th spare of component II is $A_p^{(n,m+1)}(t) - A_p^{(n,m)}(t)$. When we add (1.1) spares to (n,m) spares,

$$\begin{aligned} A_p^{(n+1,m+1)}(t) = {} & A_p^{(n,m)}(t) + [(f_1 * g_1)^{(n+1)} * \bar{F}_1]_{(t)} \\ & \cdot [1 - A_{II}^{(m)}(t)] + [(f_2 * g_2)^{(m+1)} * \bar{F}_2]_{(t)} \\ & \cdot [1 - A_I^{(n)}(t)] - [(f_1 * g_1)^{(n+1)} * \bar{F}_1]_{(t)} \\ & \cdot [(f_2 * g_2)^{(m+1)} * \bar{F}_2]_{(t)} . \qquad (4\text{-}9) \end{aligned}$$

2. Case of Sharing of Spares



Denote by $A_p^{(n)}(t)$ the availability of this parallel system when n spares are available for both components. As in IV.A.2,

$$A_p^{(0)}(t) = A_p^{\{0,0\}}(t) = 2e^{-\lambda t} - e^{-2\lambda t}$$

$$\begin{aligned} A_p^{(1)}(t) &= A_p^{\{0,0\}}(t) + A_p^{\{1,0\}}(t) + A_p^{\{0,1\}}(t) \\ &= A_p^{\{0,0\}}(t) + 2A_p^{\{1,0\}}(t) \\ &= A_p^{(0)}(t) + 2\{[f*g*\bar{F}]_{(t)} + \bar{F}(t) - (f*g*\bar{F}]_{(t)} \cdot \bar{F}(t)\} . \end{aligned}$$

In general

$$A_p^{(n+1)}(t) = A_p^{(n)}(t) + \sum_{i=0}^{(n+1)} A_p^{\{i,n+1-i\}}(t) \tag{4-10}$$

where

$$\begin{aligned} A_p^{\{i,n+1-i\}}(t) = {} & [(f*g)^{(i)}*\bar{F}]_{(t)} + [(f*g)^{(n+1-i)}*\bar{F}]_{(t)} \\ & - [(f*g)^{(i)}*\bar{F}]_{(t)} \cdot [(f*g)^{(n+1-i)}*\bar{F}]_{(t)} \end{aligned} \tag{4-11}$$

and

$$[(f*g)^{(0)}*\bar{F}]_{(t)} = \bar{F}(t)$$

or

$$A_p^{(n+1)}(t) = A_p^{(n)}(t) + 2\sum_{i=0}^{(\frac{n}{2})} A_p^{\{i,n+1-i\}}(t) \qquad (4\text{-}12)$$

if n is even

$$= A_p^{(n)}(t) + 2\sum_{i=0}^{(\frac{n-1}{2})} A_p^{\{i,n+1-i\}}(t)$$

$$+ A_p^{\{\frac{n+1}{2},\frac{n+1}{2}\}}(t) \qquad (4\text{-}13)$$

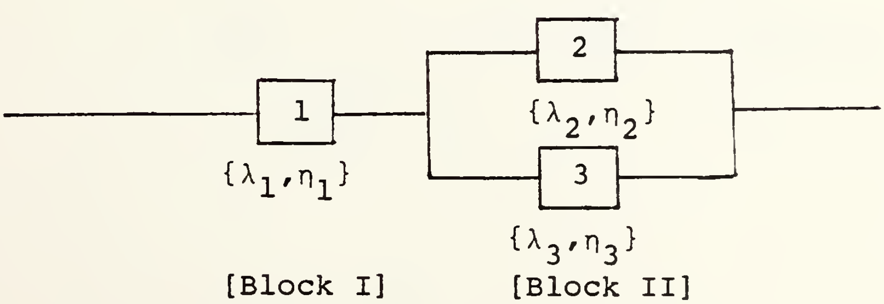
if n is odd

where

$$A_p^{\{j,k\}}(t) = [(f*g)^{(j)}*\bar{F}]_{(t)} + [(f*g)^{(k)}*\bar{F}]_{(t)}$$

$$- [(f*g)^{(j)}*\bar{F}]_{(t)} \cdot [(f*g)^{(k)}*\bar{F}]_{(t)} .$$

C. INDEPENDENT MIXED CONFIGURATION



Let $A_I^n(t)$ be the availability of Block I which is one-component subsystem consisting of component #1.

Let $A_{II}^{(m,\ell)}(t)$ be the availability of Block II which is two-component parallel subsystem consisting of components #2 and #3.

Let $A_M^{(n,m,\ell)}$ be the availability of this mixed configuration with (n,m,ℓ) spares are available. Then,

$$A_M^{(n,m,\ell)}(t) = A_I^{(n)}(t) \cdot A_{II}^{(m,\ell)}(t)$$

$$= A_{[1]}^{(n)}(t) \cdot [A_{[2]}^{(m)}(t) + A_{[3]}^{(\ell)}(t) - A_{[2]}^{(m)}(t) \cdot A_{[3]}^{(\ell)}(t)] \qquad (4\text{-}14)$$

where

$$A_{[i]}^{(k)}(t) = \text{availability at } t \text{ of component \#} i \ (i = 1,2,3).$$

When we add one more spare to one of those components, we have

$$A_M^{(n+1,m,\ell)}(t) = A_I^{(n+1)}(t) \cdot A_{II}^{(m,\ell)}(t)$$

$$= \{A_I^{(n)}(t) + [(f_1 * g_1)^{(n+1)} * \bar{F}_1]_{(t)}\} \cdot A_{II}^{(m,\ell)}(t)$$

$$= A_M^{(n.m.\ell)}(t) + [(f_1 * g_1)^{(n+1)} * \bar{F}_1]_{(t)} \cdot A_{II}^{(m,\ell)}(t) \qquad (4\text{-}15)$$

$$A_M^{(n,m+1,\ell)}(t) = A_I^{(n)}(t)\cdot\{A_{II}^{(m,\ell)}(t) + [(f_2*g_2)^{(m+1)}*\bar{F}_2]_{(t)}\}$$

$$\cdot[1 - A_{[3]}^{(\ell)}(t)]$$

$$= A_M^{(n,m,\ell)}(t) + A_I^{(n)}(t)\cdot[(f_2*g_2)^{(m+1)}*\bar{F}_2]_{(t)}$$

$$\cdot[1 - A_{[3]}^{(\ell)}(t)] \qquad (4\text{-}16)$$

$$A_M^{(n,m,\ell+1)}(t) = A_M^{(n,m,\ell)}(t) + A_I^{(n)}(t)\cdot[(f_3*g_3)^{(\ell+1)}*\bar{F}_3]_{(t)}$$

$$\cdot[1 - A_{[2]}^{(m)}(t)] \qquad (4\text{-}17)$$

Suppose component #2, and component #3 are identical such that they share p spare parts. Then

$$A_M^{(n,p)}(t) = A_I^{(n)}(t) \cdot A_{II}^{(p)}(t) . \qquad (4\text{-}18)$$

Adding one more spare to the first component

$$A_M^{(n+1,p)}(t) = \{A_I^{(n)}(t) + [(f_1*g_1)^{(n+1)}*\bar{F}_1]_{(t)}\}\cdot A_{II}^{(p)}(t)$$

$$= A_M^{(n.p)}(t) + [(f_1*g_1)^{(n+1)}*\bar{F}_1]_{(t)}\cdot A_{II}^{(p)}(t)$$

$$(4\text{-}19)$$

and adding one more spare to the second block gives

$$A_M^{(n,p+1)}(t) = A_M^{(n.p)}(t) + A_I^{(n)}(t)\cdot\Big[\sum_{i=0}^{(p+1)}\{[f_2*g_2)^{(i)}*\bar{F}2](t)$$

$$+ [(f_2*g_2)^{(p+1-i)}*\bar{F}_2]_{(t)}$$

$$- [(f_2*g_2)^{(i)}*\bar{F}_2]_{(t)}\cdot[(f_2*g_2)^{(p+1-i)}*\bar{F}_2]_{(t)}\}$$

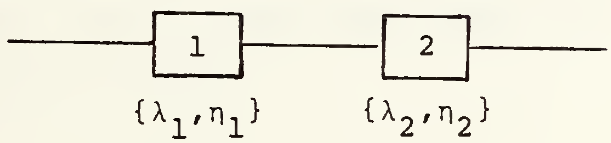
$$(4\text{-}20)$$

The marginal contribution made by the (p+1)th spare of block II is

$$A_M^{(n,p+1)}(t) - A_M^{(n,p)}(t) .$$

D. DEPENDENT TWO-COMPONENT SERIES SYSTEM FOR THE "SYMMETRIC SHUT DOWN" SCENARIO

1. Case of No Sharing of Spares



Component #1 starts its operation at time 0 with n spares and component #2 begins with m spares. The system is available if (1) the original units of components 1 and 2 survive to time t or (2) the (k,ℓ)th spares $(k=1,\ldots,n,\ \ell=1,\ldots,m)$ are alive at time t. In the case of (2) the following situations can occur. Suppose that the first failure occurs at time $s < t$ and the unit is replaced at time u $(s < u \leq t)$ by its first spare part letting the process start all over again from u with one fewer spares. Figure 4.1 shows this process for the case in which the first failure occurred in component 1.

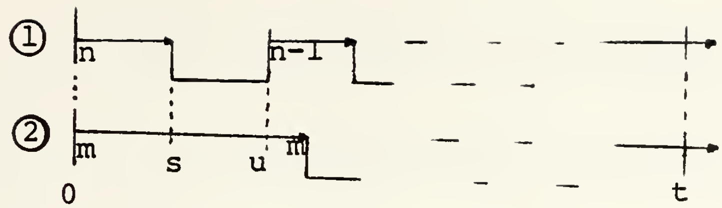


Fig. 4.1 A sample path of scenario 2

So the following recursive formula follows.

$$A_2^{(n,m)}(t) = \bar{F}_1\bar{F}_2(t) + [(f_1\bar{F}_2 * g_1) * A_2^{(n-1,m)}](t)$$

$$+ [(f_2\bar{F}_1 * g_2) * A_2^{(n,m-1)}](t) . \qquad (4\text{-}21)$$

Computational formulae can be derived by looking at the process differently. Let $A_2^{\{i,j\}}(t)$ be the availability under scenario 2 with (i,j)th spare alive at t. Looking at the first couple of terms,

$$A_2^{(1,0)}(t) = A_2^{\{0,0\}}(t) + A_2^{\{1,0\}}(t)$$

$$= \bar{F}_1\bar{F}_2(t) + [(f_1\bar{F}_2 * g_1) * \bar{F}_1\bar{F}_2](t)$$

$$A_2^{(2,0)}(t) = A_2^{(1,0)}(t) + A_2^{\{2,0\}}(t)$$

$$= A_2^{(1,0)}(t) + [(f_1\bar{F}_2 * g_1)^{(2)} * \bar{F}_1\bar{F}_2](t)$$

$$A_2^{(3,0)}(t) = A_2^{(2,0)}(t) + A_2^{\{3,0\}}(t)$$

$$= A_2^{(2,0)} + [(f_1\overline{F}_2*g_1)^{(3)}*\overline{F}_1\overline{F}_2]_{(t)}$$

$$A_2^{(1,1)}(t) = A_2^{(1,0)}(t) + A_2^{\{0,1\}}(t) + A_2^{\{1,1\}}(t)$$

$$= A_2^{(1,0)}(t) + [(f_2\overline{F}_1*g_2)*\overline{F}_1\overline{F}_2]_{(t)}$$

$$+ \binom{2}{1}[(f_1\overline{F}_2*g_1)*(f_2\overline{F}_1*g_2)*\overline{F}_1\overline{F}_2]_{(t)}$$

we have $\binom{2}{1}$ since either component can fail first.

$$A_2^{(2,1)}(t) = A_2^{(1,1)}(t) + A_2^{\{2,0\}}(t) + A_2^{\{2,1\}}(t)$$

$$= A_2^{(1,1)}(t) + [(f_1\overline{F}_2*g_1)^{(2)}*\overline{F}_1\overline{F}_2]_{(t)}$$

$$+ \binom{3}{1}[(f_1\overline{F}_2*g_1)^{(2)}*(f_2\overline{F}_1*g_2)*\overline{F}_1\overline{F}_2]_{(t)}$$

$$A_3^{(3,1)}(t) = A_2^{(2,1)}(t) + A_2^{\{3,0\}}(t) + A_2^{\{3,1\}}(t)$$

$$= A_2^{(2,1)}(t) + [(f_1\overline{F}_2*g_1)^{(3)}*\overline{F}_1\overline{F}_2]_{(t)}$$

$$+ \binom{4}{1}[(f_1\overline{F}_2*g_1)^{(3)}*(f_2\overline{F}_1*g_2)*\overline{F}_1\overline{F}_2]_{(t)} .$$

In general,

$$A_2^{(n,m)}(t) = A_2^{(n,m-1)}(t) + \sum_{i=0}^{n} \binom{m+i}{i}$$

$$\cdot [(f_1\bar{F}_2 * g_1)^{(i)} * (f_2\bar{F}_1 * g_2)^{(m)} * \bar{F}_1\bar{F}_2]_{(t)} \qquad (4\text{-}22)$$

or

$$A_2^{(n,m)}(t) = A_2^{(n-1,m)}(t)$$

$$+ \sum_{j=0}^{m} \binom{n+j}{j} \cdot [(f_1\bar{F}_2 * g_1)^{(n)} * (f_2F_1 * g_2)^{(j)} * \bar{F}_1\bar{F}_2]_{(t)} \qquad (4\text{-}23)$$

where

$$[h^{(0)} * \bar{F}]_{(\cdot)} = \bar{F}_{(\cdot)} .$$

Either of eq. (4-22) or eq. (4-23) is preferable to eq. (4-21) for computational purposes.

To evaluate eq. (4-22) or eq. (4-23) we need to evaluate the term $[(f_1\bar{F}_2 * g_1)^{(i)} * (f_2\bar{F}_1 * g_2)^{(j)} * \bar{F}_1\bar{F}_2]_{(t)}$.

$$\mathcal{L}\{[(f_1\bar{F}_2 * g_1)^{(i)} * (f_2\bar{F}_1 * g_2)^{(j)} * \bar{F}_1\bar{F}_2]_{(t)}\}$$

$$= \frac{(\lambda_1\eta_1)^i(\lambda_2\eta_2)^j}{(s+\lambda_1+\lambda_2)^{i+j+1}(s+\eta_1)^i(s+\eta_2)^j} \qquad (4\text{-}24)$$

For $(i,j) = (1,1)$ we have

$$\frac{(\lambda_1\eta_1)(\lambda_2\eta_2)}{(s+\lambda_1+\lambda_2)^3(s+\eta_1)(s+\eta_2)} = (\lambda_1\eta_1)(\lambda_2\eta_2)\Big[\frac{A}{(s+\lambda_1+\lambda_2)} + \frac{B}{(s+\lambda_1+\lambda_2)^2} + \frac{C}{(s+\lambda_1+\lambda_2)^3} + \frac{D}{(s+\eta_1)} + \frac{E}{(s+\eta_2)}\Big]$$

Let $a = \eta_1 - \lambda_1 - \lambda_2$, $b = \eta_2 - \lambda_1 - \lambda_2$, then

$$A = \frac{a^2 + b^2 + ab}{a^3b^3}, \quad B = -\frac{a+b}{a^2b^2}, \quad C = \frac{1}{ab}$$

$$D = -\frac{1}{a^3(\eta_2-\eta_1)}, \quad E = \frac{1}{b^3(\eta_2-\eta_1)}$$

Thus,

$$[(f_1\overline{F}_2*g_1)*(f_2\overline{F}_1*g_2)*\overline{F}_1\overline{F}_2]_{(t)}$$

$$= (\lambda_1\eta_1)(\lambda_2\eta_2)\Big[\frac{a^2+b^2+ab}{a^3b^3} e^{-(\lambda_1+\lambda_2)t} - \frac{a+b}{a^2b^2} te^{-(\lambda_1+\lambda_2)t} + \frac{1}{ab}\cdot\frac{t^2}{2} e^{-(\lambda_1+\lambda_2)t} - \frac{1}{a^3(\eta_2-\eta_1)} e^{-\eta_1 t} + \frac{1}{b^3(\eta_2-\eta_1)} e^{-\eta_2 t}\Big] .$$

As another example consider the case $(i,j) = (2,1)$. Then

$$[(f_1\bar{F}_2*g_1)^{(2)}*(f_2\bar{F}_1*g_2)*\bar{F}_1\bar{F}_2]_{(t)}$$

$$= (\lambda_1\eta_1)^2(\lambda_2\eta_2)\{Ae^{-(\lambda_1+\lambda_2)t} + Bte^{-(\lambda_1+\lambda_2)t}$$

$$+ C\cdot\frac{t^2}{2}e^{-(\lambda_1+\lambda_2)t} + D\cdot\frac{t^3}{3!}e^{-(\lambda_1+\lambda_2)t}$$

$$+ E\cdot e^{-\eta_1 t} + Fte^{-\eta_1 t} + Ge^{-\eta_2 t}\}$$

where

$$A = \frac{1}{3!}\cdot\frac{(4b+3a)\{(3b+2a)(2b+a)-3ab\}-ab\{5(2b+a)+3(3b+2a)-3(a+b)\}}{a^5b^4}$$

$$B = \frac{1}{2}\cdot\frac{(3b+2a)(2b+a)-3ab}{a^4b^3}$$

$$C = -\frac{2b+a}{a^3b^2}$$

$$D = \frac{1}{a^2b}$$

$$E = -\frac{4(\eta_2-\eta_1)-a}{a^5(\eta_2-\eta_1)^2}$$

$$F = \frac{1}{a^4(\eta_2-\eta_1)}$$

$$G = \frac{1}{a^4(\eta_2-\eta_1)^2}$$

and $a = \eta_1-\lambda_1-\lambda_2$, $b = \eta_2-\lambda_1-\lambda_2$.

As (i,j) gets larger it becomes more difficult to invert the Laplace transform. In these cases it is often reasonable to make some assumptions which simplify the expressions.

In many real situations the replacement rates η_i are very large compared to the failure rates λ_i. When this happens the expressions involving $e^{-\eta_i t}$ are dominated by the expressions involving $e^{-\lambda_i t}$ and good approximations for the availability can be obtained by ignoring the terms involving $e^{-\eta_i t}$. The formulae can be simplified significantly without setting $e^{-\eta_i t}$ to zero if a single replacement rate $\eta_1 = \eta_2 = \eta$ can be assumed. The expression for availability using the common replacement rate η is much more tractable.

It can be shown that using $\max\{\eta_i\}$ for the common replacement rate will give an upper bound of the true availability and $\min\{\eta_i\}$ a lower bound. A weighted replacement rate may be used for a good approximation too.

In any case, let the common replacement rate be η, then eq. (4-24) becomes

$$\mathcal{L}\{[f_1\bar{F}_2*g)^{(i)}*(f_2\bar{F}_1*g)^{(j)}*\bar{F}_1\bar{F}_2]_{(t)}\}$$

$$= \frac{\lambda_1^i\lambda_2^j\eta^{i+j}}{(s+\lambda_1+\lambda_2)^{i+j+1}(s+\eta)^{i+j}} \tag{4-25}$$

Taking the inverse transformation gives

$$[(f_1\bar{F}_2^{*}g)^{(i)} * (f_2\bar{F}_1 * g)^{(j)} * \bar{F}_1\bar{F}_2]_{(t)}$$

$$= \frac{\lambda_1^i\lambda_2^j\eta^{i+j}}{\delta^{i+j}} \cdot \left[\frac{t^{i+j}}{(i+j)!} + \sum_{r=1}^{i+j}(-1)^r \cdot \frac{{}_{(i+j+r-1)}P_r}{r!\delta^r} \cdot \frac{t^{i+j-r}}{(i+j-r)!}\right] e^{-(\lambda_1+\lambda_2)t}$$

$$+ (-1)^{i+j+1} \cdot \frac{\lambda_1^i\lambda_2^j\eta^{i+j}}{\delta^{i+j+1}} \left[\frac{t^{i+j-1}}{(i+j-1)!} + \sum_{k=1}^{i+j-1} \frac{{}_{(i+j+k)}P_k}{k!\delta^k} \cdot \frac{t^{i+j-k-1}}{(i+j-k-1)!}\right] e^{-\eta t} \qquad (4\text{-}26)$$

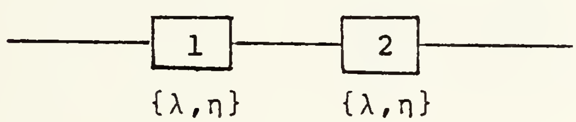
where

$$\delta = \eta - \lambda_1 - \lambda 2$$

and

$$_nP_k = n(n-1)\ldots(n-k+1) = \frac{n!}{(n-k)!}$$

2. <u>Case of Sharing of Spares</u>



Using the same notations as in IV.A.2,

$$A_2^{(1)}(t) = A_2^{\{0,0\}}(t) + 2A_2^{\{1,0\}}(t)$$

$$= A_2^{(0)}(t) + 2[f\bar{F}*g*\bar{F}^2]_{(t)}$$

$$A_2^{(2)}(t) = A_2^{(1)}(t) + 2A_2^{\{2,0\}}(t) + A_2^{\{1.1\}}(t)$$

$$= A_2^{(1)}(t) + (2+2)[(f\bar{F}*g)^{(2)}*\bar{F}^2]_{(t)}$$

$$A^{(3)}(t) = A_2^{(2)}(t) + 2A_2^{\{3,0\}}(t) + 2A_2^{\{2,1\}}(t)$$

$$= A_2^{(2)}(t) + [2 + 2\binom{3}{1}]\cdot[(f\bar{F}*g)^{(3)}*\bar{F}^2]_{(t)}$$

and in general

$$A_2^{(n)}(t) = A_2^{(n-1)}(t) + 2^n[(f\bar{F}*g)^{(n)}*\bar{F}^2]_{(t)} \tag{4-27}$$

which can also be obtained from (3-15). And

$$[(f\bar{F}*g)^{(n)}*\bar{F}^2]_{(t)} = [\frac{\theta}{\delta}]^n [\frac{t^n}{n!} + \sum_{r=1}^{n} (-1)^r \frac{(n+r-1)^{P_r}}{r!\delta^r}\cdot \frac{t^{n-r}}{(n-r)!}]e^{-2\lambda t}$$

$$+ (-1)^{n+1}[\frac{\theta^n}{\delta^{n+1}}][\frac{t^{n-1}}{(n-1)!}$$

$$+ \sum_{\ell=1}^{n-1} \frac{(n+\ell)^{P_k}}{\ell!\delta^\ell}\cdot\frac{t^{n-\ell-1}}{(n-\ell-1)!}]e^{-\eta t} \tag{4-28}$$

where

$$\theta = \eta\lambda ,$$

$$\delta = \eta - 2\lambda .$$

E. A NUMERICAL EXAMPLE

Consider a two-component series system. We assume for simplicity that the components are identical. We consider two cases: 1) spares are shared by both components, and

2) each component has a separate spares pool with no sharing allowed. With the parameters used in the previous example, $\lambda = \frac{1}{30}$, $\eta = \frac{1}{5}$, and $t = 90$, we obtain the results shown in Table 4-1 from equations (4-5), (4-28), (2-2), (2-3), and (4-26).

Table 4-1: Computation Results

Spares Shared			Spares not Shared			
n	Scenario 1	Scenario 2	(n_1,n_2)	Scenario 1	Scenario 2	Scenario 3*
0	0.00248	0.00248	(0,0)	0.00248	0.0024788	0.00248
1	0.01914	0.02293	(1,0)	0.01081	0.012704	0.0.081
2	0.07154	0.09404	(0,1)	0.01081	0.012404	0.01270
3	0.17750	0.23416	(1,1)	0.04713	0.05848	0.05786
4	0.31973	0.41220	(2,1)	0.10036	0.1288	
5	0.46724	0.56961	(1,2)	0.10036	0.1288	
6	0.58768	0.67178	(2,2)	0.21371	0.2659	
7	0.66670	0.72204	(3,2)	0.31074	0.4208	
			(2,3)	0.31074	0.4208	
			(3,3)	0.45180	0.5668	

The following computations show how the result for $A_3^{(1,1)}(t)$ was obtained.

$$A_3^{\{0,0\}}(t) = \overline{F}^2(t)$$

$$A_3^{\{1,0\}}(t) = [f*g*\overline{F}]_{(t)} \cdot \overline{F}(t)$$

$$A_3^{\{0,1\}}(t) = [f\overline{F}*g*\overline{F}^2]_{(t)}$$

$$A_3^{\{1,1\}}(t) = 2[(f\overline{F})^{(2)}*g*g\overline{F}*\overline{F}^2]_{(t)} + [f\overline{F}*f\overline{G}*g\overline{F}*g\overline{G}*\overline{F}^2]_{(t)}$$

$$+ [f\overline{F}*f\overline{G}*g\overline{G}*g*\overline{F}^2]_{(t)}$$

and

$$2[(f\overline{F})^{(2)}*g*g\overline{F}*\overline{F}^2]_{(t)} = 2(\eta\lambda)^2[A + Bt + \frac{C}{2}t^2)e^{-2\lambda t}$$

$$+ De^{-(\eta+\lambda)t} + Ee^{-\eta t}]$$

where

$$A = \frac{(2\eta-3\lambda)^2 - (\eta-\lambda)(\eta-2\lambda)}{(\eta-2\lambda)^3(\eta-\lambda)^3},$$

$$B = -\frac{(2\eta-3\lambda)}{(\eta-2\lambda)^2(\eta-\lambda)^2},$$

$$C = \frac{1}{(\eta-2\lambda)(\eta-\lambda)},$$

$$D = \frac{1}{(\eta-\lambda)^3\lambda},$$

$$E = -\frac{1}{(\eta-2\lambda)^3\lambda}$$

$$[f\overline{F}*f\overline{G}*g\overline{F}*g\overline{G}*\overline{F}^2]_{(t)} = (\eta\lambda)^2[(-\frac{5}{4(\eta-\lambda)^4} + \frac{t}{2(\eta-\lambda)^4})e^{-2\lambda t}$$

$$+ (\frac{1}{(\eta-\lambda)^4} + \frac{t}{(\eta-\lambda)^3})e^{-(\eta+\lambda)t} + \frac{1}{4(\eta-\lambda)^6}e^{-2\eta t}],$$

and

$$[f\overline{F}*f\overline{G}*g\overline{G}*g*\overline{F}^2]_{(t)} = (\eta\lambda)^2[(-\frac{5}{4(\eta-\lambda)^3(\eta-2\lambda)} + \frac{t}{2(\eta-\lambda)^2(\eta-2\lambda)})e^{-2\lambda t}$$

$$- \frac{1}{(\eta-\lambda)^3\lambda} e^{-(\eta+\lambda)t} + \frac{1}{\eta\lambda(\eta-2\lambda)^2} e^{-\eta t} + \frac{1}{4\eta(\eta-\lambda)^3} e^{-2\eta t}]$$

Thus we have

$$\begin{aligned} A_3^{(1,1)}(t) &= A_3^{\{0,0\}}(t) + A_3^{\{1,0\}}(t) + A_3^{\{0,1\}}(t) + A_3^{\{1,1\}}(t) \\ &= 0.00248 + 0.00833 + 0.01022 + 0.03683 \\ &= 0.05786 . \end{aligned}$$

V. COMPARISONS OF POINT AVAILABILITIES FOR DIFFERENT SCENARIOS

In this chapter we continue to consider the point availabilities of two-component series systems which operate in accordance with scenarios one through five. Because of the complications that arise when trying to obtain explicit mathematical formulae for the point availabilities in some of the cases examined earlier, we attempt in this chapter to obtain order relationships among the point availabilities of the five scenarios. Such order relationships allow bounds to be placed on the point availabilities for the scenarios that generate the most computational problems. Intuitively, we would expect that the scenario point availabilities would satisfy:

$$A_1^{(n,m)}(t) \leq A_3^{(n,m)}(t) \leq A_2^{(n,m)}(t)$$

since the surviving component remains in operation in scenario one (and is therefore subject to failure) while a failed component is being replaced; the surviving component remains in operation sometimes in scenario three (depending on which component failed); and the surviving component never remains in service in scenario two. Intuitively, the more the components are in service, the more they should fail and the lower would be the system point availability. Similarly, one would expect that

$$A_1^{(n,m)}(t) \le A_4^{(n,m)}(t) \le A_2^{(n,m)}(t)$$

and

$$A_5^{(n,m)}(t) \le A_1^{(n,m)}(t)$$

This chapter attempts to prove these relationships.

A. COMPARISONS BETWEEN SCENARIOS 1, 2, 3

In this section we consider comparisons among the point availabilities for two-component series systems described by scenarios one, two, and three.

Let us first develop a few results that will be useful in deriving the order relationships among the point availabilities. Consider a one-component system with n spares available. Let $A^{(n)}(t)$ be the availability of this system given that the system started up at time 0 with n spares. Let $B^{(n)}(t)$ be the availability at time t given that the system started down at time 0 with n spares available. The following corollary proves that A(t;n) is at least as large as B(t;n) for all t > 0 and n. That is, the probability that the system will be found in an up state at time t, is greater whenever the system begins in an up state than if the system begins in a down state.

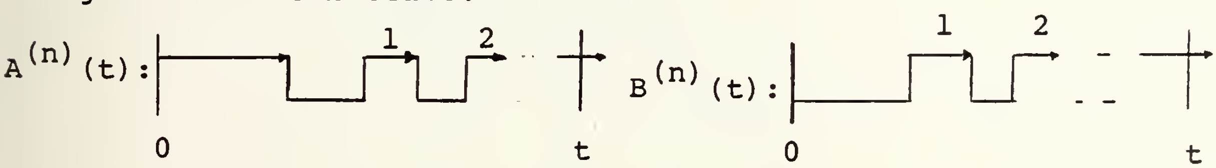


Fig. 5.1 Two Processes for $A^{(n)}(t)$ and $B^{(n)}(t)$

Corollary 5-1

$$A^{(n)}(t) \geq B^{(n)}(t) \qquad t \geq 0, \quad n = 0,1,2,...$$

Proof:

Let s be the time of the first failure of the component. The system will be operational at time t provided either $s > t$ or $s < t$ and the system is resurrected and operational at time t. In the latter case, at time s, the probability of being up at t is $B^{(n)}(t-s)$.

$$A^{(n)}(t) = \bar{F}(t) + \int_0^t f(s)B^{(n)}(t-s)ds \tag{5-1}$$

Similarly, for $B^{(n)}(t)$ the first transition (replacement) must occur at some time $s < t$ at which point the probability of being up at time t is $A^{(n-1)}(t-s)$. Thus,

$$B^{(n)}(t) = \int_0^t g(s)A^{(n-1)}(t-s)ds \ . \tag{5-2}$$

In order to show that $A^{(n)}(t) \geq B^{(n)}(t)$ we re-express (5-1) and (5-2) as follows:

$$A^{(n)}(t) = \bar{F}(t) + \int_0^t f\bar{G}(s)B^{(n)}(t-s)ds + \int_0^t g\bar{F}(s)A^{(n)}(t-s)ds$$

$$- \int_0^t g\bar{F}(s)\bar{F}(t-s)ds \tag{5-3}$$

and

$$B^{(n)}(t) = \int_0^t f\bar{G}(s)B^{(n)}(t-s)ds + \int_0^t g\bar{F}(s)A^{(n-1)}(t-s)ds \tag{5-4}$$

Equations (5-3) and (5-4) were determined heuristically. Verification of the equations is provided by showing (5-3) to be equivalent to (5-1) and (5-4) to be equivalent to (5-2). We do this by showing the Laplace transforms to be identical. By the uniqueness theorem of Laplace transforms this proves that the expressions are equivalent.

The Laplace transform of (5-3) is given by

$$A^{(n)*}(s) = \frac{1}{s+\lambda} + \frac{\lambda}{s+\lambda+\eta} B^{(n)*}(s) + \frac{\eta}{s+\lambda+\eta} A^{(n)*}(s) - \frac{\eta}{s+\lambda+\eta}\cdot\frac{1}{s+\lambda}$$

which simplifies to be

$$A^{(n)*}(s) = \frac{1}{s+\lambda} + \frac{\lambda}{s+\lambda} B^{(n)*}(s) ,$$

which is the Laplace transform of (5-1).

Similarly, the Laplace transform of (5-4) is given by

$$B^{(n)*}(s) = \frac{\lambda}{s+\lambda+\eta}B^{(n)*}(s) + \frac{\eta}{s+\lambda+\eta}A^{(n-1)*}(s)$$

$$= \frac{\eta}{s+\lambda} A^{(n-1)*}(s)$$

which is the Laplace transform of (5-2). Thus (5-1) is equivalent to (5-3) and (5-2) is equivalent to (5-4).

Now, in (5-3)

$$\bar{F}(t) - \int_0^t g\bar{F}(s)\bar{F}(t-s)ds = e^{-(\eta+\lambda)t} > 0$$

and from (2-1)

$$A^{(n)}(t) \geq A^{(n-1)}(t) \quad \text{for} \quad t \geq 0$$

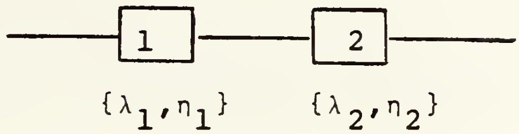
Thus,

$$A^{(n)}(t) = \int_0^t f\bar{G}(s)B^{(n)}(t-s) + \int_0^t g\bar{F}(s)A^{(n)}(t-s)ds + e^{-(\lambda+\eta)t}$$

$$\geq \int_0^t f\bar{G}(s)B^{(n)}(t-s) + \int_0^t g\bar{F}(s)A^{(n-1)}(t-s)$$

$$= B^{(n)}(t).$$

Now consider a two-component series system operating under different operating scenarios.



Denote by $A_i^{(n,m)}(t)$ the availability of this system operating under scenario i when the first component starts to operate from the "up" state with n spares available and the second component starts to operate from the "up" state with m spares available. Then the following theorem holds.

Theorem 5.2.

$$A_1^{(n,m)}(t) \leq A_3^{(n,m)}(t) \leq A_2^{(n,m)}(t)$$

$$t \geq 0, \quad n,m = 0,1,2,3, \ldots$$

The proof of Theorem 5-2 is inductive. First consider $(n,m) = (0,0)$ and note that

$$A_1^{(0,0)}(t) = A_2^{(0,0)}(t) = A_3^{(0,0)}(t) = e^{-(\lambda_1+\lambda_2)t} .$$

Next note that for $(n,m) = (1,0)$

$$A_1^{(1,0)}(t) = [\bar{F}_1(t) + (f_1*g_1*\bar{F}_1)_{(t)}]\bar{F}_2(t)$$

$$= \bar{F}_1\bar{F}_2(t) + [f_1\bar{F}_2*g_1\bar{F}_2*\bar{F}_1\bar{F}_2]_{(t)}$$

$$A_2^{(1,0)}(t) = \bar{F}_1\bar{F}_2(t) + [f_1\bar{F}_2 * g_1 * \bar{F}_1\bar{F}_2](t)$$

$$A_3^{(1,0)}(t) = \bar{F}_1\bar{F}_2(t) + [f_1\bar{F}_2 * g_1\bar{F}_2 * \bar{F}_1\bar{F}_2](t)$$

since

$$\bar{F}(\cdot) \leq 1$$

$$A_1^{(1,0)}(t) = A_3^{(1,0)}(t) \leq A_2^{(1,0)}(t) .$$

Likewise, we have for $(n,m) = (0,1)$ the relationships

$$A_1^{(0,1)}(t) = \bar{F}_1\bar{F}_2(t) + [f_2\bar{F}_1 * g_2\bar{F}_1 * \bar{F}_1\bar{F}_2](t)$$

$$A_2^{(0,1)}(t) = \bar{F}_1\bar{F}_2(t) + [f_2\bar{F}_1 * g_2 * \bar{F}_1\bar{F}_2](t)$$

$$A_3^{(0,1)}(t) = \bar{F}_1\bar{F}_2(t) + [f_2\bar{F}_1 * g_2 * \bar{F}_1\bar{F}_2](t)$$

and

$$A_1^{(0,1)}(t) \leq A_3^{(0,1)}(t) = A_2^{(0,1)}(t) .$$

Assume that

$$A_1^{(i,j)}(t) \leq A_3^{(i,j)}(3) \leq A_2^{(i,j)}(t) , \quad t \geq 0$$

and for all (i,j) such that $i < n$, $j < m$, then we will show that

$$A_1^{(n,m)}(t) \leq A_3^{(n,m)}(t) \leq A_2^{(n,m)}(t) , \quad t \geq 0 .$$

Before continuing with the proof of Theorem 5-2 we define the following random variables:

T_j : Time to failure of the first unit of component j

R_j : Replacement time of the first unit of component j

$X(t)$: State of the process at t (0 or 1)

$V_j(\tau)$: State of the component j at time τ (0 or 1)

We also let $A_{D_i}^{(k,\ell)}$ be the availability at time t starting from state D_i; $(i = 1,2)$ with (k,ℓ) spares available, and let $T = \min(T_1,T_2)$, $T_1 + R_1 = S_1$, and $T_2 + R_2 = S_2$. Furthermore, we define the following seven cases:

Case 1: $T = T_1 = u_1$, $0 < u_1 < s_1 \leq t$ and component 2 has k failures in $(u_1,s_1]$ with the kth spare alive at s_1. So at s_1 the system is in state A with $(n-1,m-k)$ spares available (Fig. 5-2).

Case 2: $T = T_1 = u_1$, $0 < u_1 < s_1 \leq t$ and component 2 has $(k+1)$ failures in $(u_1,s_1]$ with the $(k+1)$th replacement underway at s_1. So at s_1 the system is in state D_2 with $(n-1,m-k)$ spares available (Fig. 5-3).

Case 3: $T = T_1 = u_1, 0 < u_1 < s_1 \leq t$ and component 2 has no failures in $(u_1,s_1]$ (Fig. 5-4).

Case 4, 5 and 6 are the symmetric analysis of cases 1, 2 and 3, respectively obtained by changing the roles of the two components (Figs. 5-5, 5-6, 5-7).

Let $p_1(s,k)$ be the probability of the event taking place over the interval $(0,s_1]$ described in case 1. Similarly, let $p_2(s_1,k)$, $p_3(s_1)$, $p_4(s_2,k)$, $p_5(s_2,\ell)$, and $p_6(s_2)$ be the

Case 7: $T > t$.

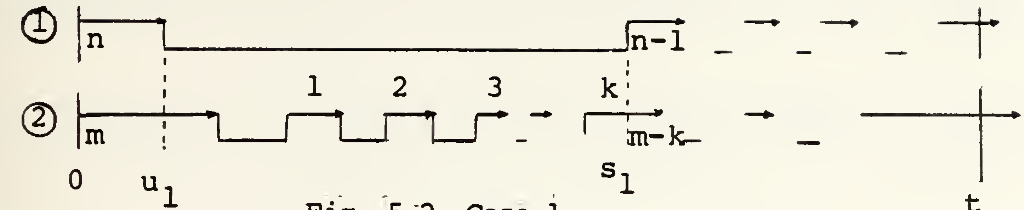


Fig. 5.2 Case 1

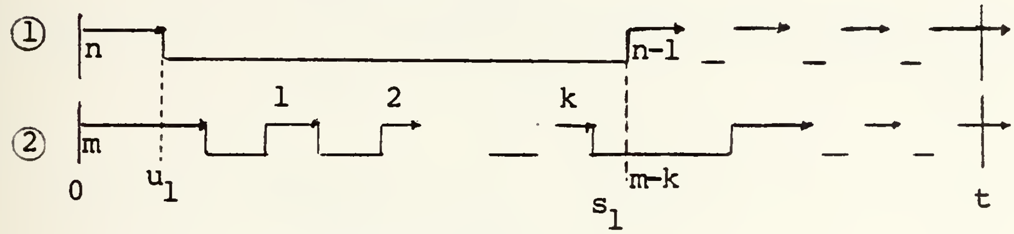


Fig. 5.3 Case 2

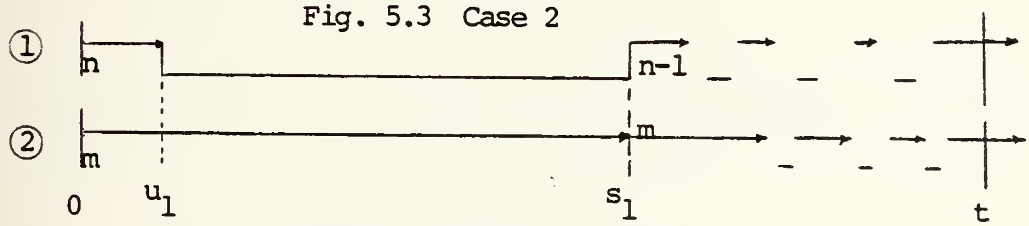


Fig. 5.4 Case 3

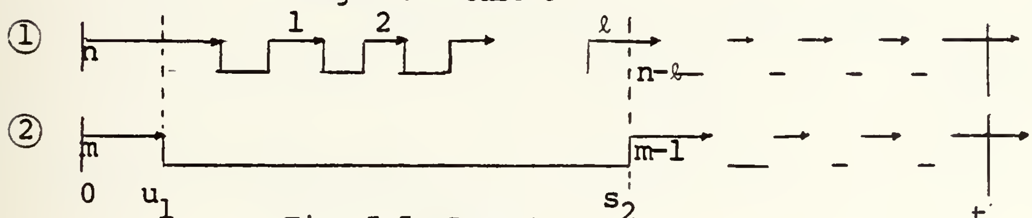


Fig. 5.5 Case 4

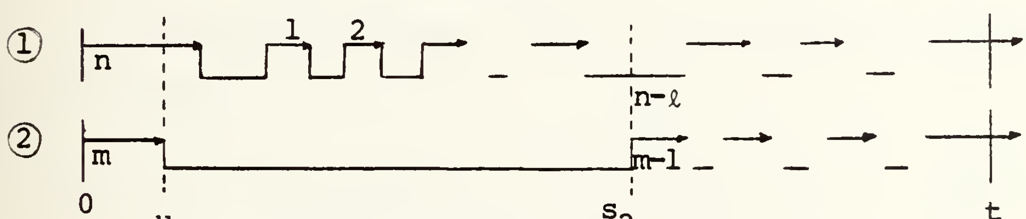


Fig. 5.6 Case 5

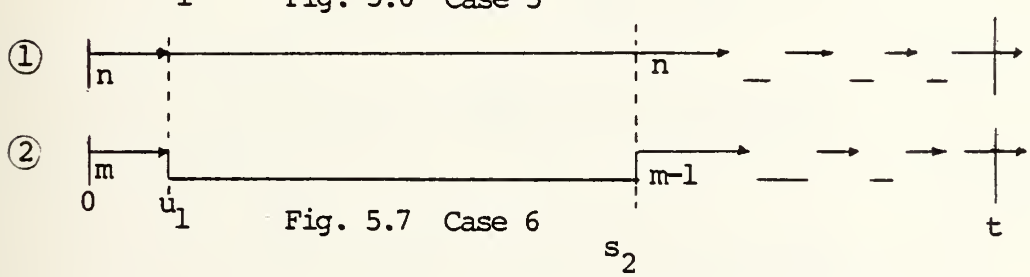


Fig. 5.7 Case 6

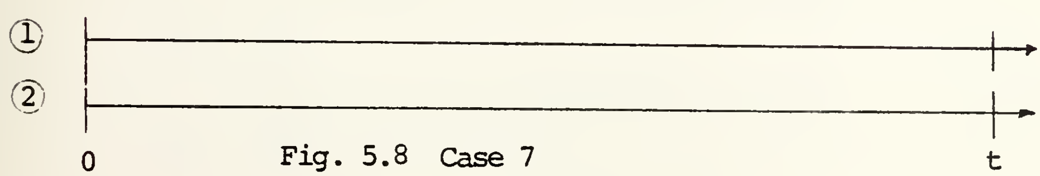


Fig. 5.8 Case 7

probabilities of the events taking place in $(0,s_i]$ described in cases 2 through 6. From the figures, we see that:

$$p_1(s_1,k) = \int_{u=0}^{s_1} [f_1\bar{F}_2(u)\{g_1[(f_2*g_2)^{(k)}*\bar{F}_2]\}(s_1-u)du$$

$$p_2(s_1,k) = \int_{u=0}^{s_1} [f_1\bar{F}_2(u)\{g_1[f_2*g_2)^{(k)}*f_2*\bar{G}_2]\}(s_1-u)du$$

$$p_3(s_1) = [f_1\bar{F}_2*g_1](s_1)$$

$$p_4(s_2,\ell) = \int_{u=0}^{s_2} [f_2\bar{F}_1(u)\{g_2[(f_1*g_1)^{(\ell)}*\bar{F}_1]\}(s_2-u)du$$

$$p_5(s_2,\ell) = \int_{u=0}^{s_2} [f_2\bar{F}_1(u)\{g_2[(f_1*g_1)^{(\ell)}*f_1*\bar{G}_1]\}(s_2-u)du$$

$$p_6(s_2) = [f_2\bar{F}_1*g_2](s_2)$$

For scenario 1, cases 3 and 6 are included in cases 2 and 5; therefore,

$$\begin{aligned}
A_1^{(n,m)}(t) &= \bar{F}_1(t)\bar{F}_2(t) + \int_{s_1=0}^{t} \sum_{k=0}^{m-1} p_1(s_1,k)\cdot A_1^{(n-1,m-k)}(t-s_1)ds_1 \\
&+ \int_{s_1=0}^{t} \sum_{k=0}^{m-1} p_2(s_1,k)\cdot A_{1D_2}^{(n-1,m-k)}(t-s_1)ds_1 \\
&+ \int_{s_2=0}^{t} \sum_{\ell=0}^{n-1} p_4(s_2,\ell)\cdot A_1^{(n-\ell,m-1)}(t-s_2)ds_2 \\
&+ \int_{s_2=0}^{t} \sum_{\ell=0}^{n-1} p_5(s_2,\ell)\cdot A_{1D_1}^{(n-\ell,m-1)}(t-s_2)ds_2 \qquad (5\text{-}5)
\end{aligned}$$

Likewise, for scenario 2, cases 1, 2, 4 and 5 are not possible; therefore,

$$A_2^{(n,m)}(t) = \bar{F}_1(t)\bar{F}_2(t) + \int_0^t p_3(s_1)\cdot A_2^{(n-1,m)}(t-s_1)ds_1$$

$$+ \int_0^t p_6(s_2)\cdot A_2^{(n,m-1)}(t-s_2)ds_2 \qquad (5\text{-}6)$$

For scenario 3, cases 4 and 5 are not possible and case 3 is contained in case 2. We obtain

$$A_3^{(n,m)}(t) = \bar{F}_1(t)\bar{F}_2(t) + \int_0^t \sum_{k=0}^{m-1} p_1(s_1,k)\cdot A_3^{(n-1,m-k)}(t-s_1)ds_1$$

$$+ \int_0^t \sum_{k=0}^{m-1} p_2(s_1,k)\cdot A_{3D_2}^{(n-1,m-k)}(t-s_1)ds_1$$

$$+ \int_0^t p_6(s_2)\cdot A_3^{(n,m-1)}(t-s_2)ds_2 \qquad (5\text{-}7)$$

Let us now compare $A_1^{(n,m)}(t)$, $A_2^{(n,m)}(t)$, and $A_3^{(n,m)}(t)$ using (5-5), (5-6), and (5-7). We have,

$$A_1^{(n,m)}(t) \le \bar{F}_1\bar{F}_2 + \int_0^t \sum_{k=0}^{m-1} P_1(s_1,k)A_1^{(n-1,m-k)}(t-s_1)ds_1$$

$$+ \int_0^t \sum_{k=0}^{m-1} P_2(s_1,k)A_{1D_2}^{(n-1,m-k)}(t-s_1)ds_1$$

$$+ \int_0^t \sum_{\ell=0}^{n-1} P_4(s_2,\ell)A_1^{(n-\ell,m-1)}(t-s_2)ds_2$$

$$+ \int_0^t \sum_{\ell=0}^{n-1} P_5(s_2,\ell)A_1^{(n-\ell,m-1)}(t-s_2)ds_2$$

$$\le \bar{F}_1\bar{F}_2 + \int_0^t \sum_{k=0}^{m-1} P_1(s_1,k)A_1^{(n-1,m-k)}(t-s_1)ds_1$$

$$+ \int_0^t \sum_{k=0}^{m-1} P_2(s_1,k)A_{1D_2}^{(n-1,m-k)}(t-s_1)ds_1$$

$$+ \int_0^t \sum_{\ell=0}^{n-1} P_4(s_2,\ell)A_1^{(n,m-1)}(t-s_2)ds_2$$

$$+ \int_0^t \sum_{\ell=0}^{n-1} P_5(s_2,\ell)A_1^{(n,m-1)}(t-s_2)ds_2$$

$$\le \bar{F}_1\bar{F}_2 + \int_0^t \sum_{k=0}^{m-1} P_1(s_1,k)A_1^{(n-1,m-k)}(t-s_1)ds_1$$

$$+ \int_0^t \sum_{k=0}^{m-1} P_2(s_1,k)A_{1D_2}^{(n-1,m-k)}(t-s_1)ds_1$$

$$+ \int_0^t [f_2\bar{F}_1 * g_2]_{(s_2)} \cdot A_1^{(n,m-1)}(t-s_1)ds_1$$

$$\le \bar{F}_1\bar{F}_2 + \int_0^t \sum_{k=0}^{m-1} P_1(s_1,k)A_1^{(n-1,m-k)}(t-s_2)ds_2$$

$$+ \int_0^t \sum_{k=0}^{m-1} P_2(s_1,k) A_{1D_2}^{(n-1,m-k)}(t-s_1)ds_1$$

$$+ \int_{s_2=0}^{t} [f_2\bar{F}_1 * g_2]_{(s_2)} \cdot A_3^{(n,m-1)}(t-s_1)ds_1$$

$$= A_3^{(n,m)}(t) .$$

The first inequality comes from

$$A_1^{(i,j)}(t) \geq A_{1D_j}^{(i,j)}(t)$$

which is immediate result of corollary 5-1. The second inequality comes from

$$A_1^{(i,j)}(\tau) > A_1^{(k,\ell)}(\tau)$$

for $(i,j) > (k,\ell)$ and $\tau > 0$. The third inequality comes from the fact that

$$\sum_{\ell=0}^{n-1} [(f_1 * g_1)^{(\ell)} * (\bar{F}_1 + f_1 * \bar{G}_1)]_{(\tau)} = P\{\text{Number of complete cycles for component 1 in } (0,\tau) \leq n-1\} \leq 1$$

for $n = 0,1,2,3,\ldots$., and the last inequality comes from the assumption that

$$A_1^{(i,j)}(\tau) \leq A_3^{(i,j)}(\tau)$$

for $(i,j) < (n,m)$ and $\tau \geq 0$. It is easy to see from the same

facts used above that

$$A_3^{(n,m)}(t) \leq \bar{F}_1\bar{F}_2(t) + [f_1\bar{F}_2 * g_1 * A_3^{(n-1,m)}]_{(t)}$$

$$+ [f_2\bar{F}_1 * g_2 A_3^{(n,m-1)}]_{(t)}$$

$$\leq \bar{F}_1\bar{F}_2 + [f_1\bar{F}_2 * g_1 * A_2^{(n-1,m)}]_{(t)}$$

$$+ [f_2\bar{F}_1 * g_2 * A_2^{(n,m-1)}]_{(t)} = A_2^{(n,m)}(t) .$$

This completes the proof of Theorem 5-2.

B. COMPARISONS BETWEEN SCENARIOS 1, 4 AND 5

The following corollaries are useful in comparing scenarios 1, 4 and 5.

Corollary 5-3.

Consider two single-component systems. Let the first have exponential iid lifetimes with parameter λ_1 and exponential iid replacement time with parameter η. Similarly, let the second have exponential iid lifetimes and replacement times with parameters λ_2 and η, respectively. If $\lambda_2 \geq \lambda_1$, then

$$\frac{\bar{F}_1(t)}{\bar{F}_1(t)+[f_1 * \bar{G}]_{(t)}} \geq \frac{\bar{F}_2(t)}{\bar{F}_2(t)+[f_2 * \bar{G}]_{(t)}}$$

Proof:

$$\frac{\bar{F}_i(t)}{[f_i*\bar{G}]_{(t)}} = \frac{e^{-\lambda_i t}}{\frac{\lambda_i}{\eta-\lambda_i}[e^{-\lambda_i t}-e^{-\eta t}]} = \frac{1}{\frac{\lambda_i}{\eta-\lambda_i}[1-e^{-(\eta-\lambda_i)t}]}$$

Now, let

$$\ell(t) = \frac{\lambda_2}{\eta-\lambda_2}[1-e^{-(\eta-\lambda_2)t}] - \frac{\lambda_1}{\eta-\lambda_1}[1-e^{-(\eta-\lambda_1)t}]$$

$$= \frac{\lambda_2}{\eta-\lambda_2} - \frac{\lambda 1}{\eta-\lambda_1} - [\frac{\lambda 2}{\eta-\lambda_2}e^{-(\eta-\lambda_2)t} - \frac{\lambda_1}{\eta-\lambda_1}e^{-(\eta-\lambda_1)t}]$$

Taking the derivative wrt t, we see that

$$\frac{d}{dt}\ell(t) = \lambda_2 e^{-(\eta-\lambda_2)} - \lambda_1 e^{-(\eta-\lambda_1)t} \geq 0$$

since $\lambda_2 \geq \lambda_1$.

Since $\ell(0) = 0$ and $\ell(t)$ is increasing in t, $\ell(t) \geq 0$, $t \geq 0$. Thus

$$\frac{\bar{F}_1(t)}{[f_1*\bar{G}]_{(t)}} \geq \frac{\bar{F}_2(t)}{[f_2*\bar{G}]_{(t)}}$$

and

$$\frac{\bar{F}_1(t)}{\bar{F}_1(t)+[f_1*\bar{G}]_{(t)}} \geq \frac{\bar{F}_2(t)}{\bar{F}_2(t)+[f_2*\bar{G}]_{(t)}} \qquad \text{Q.E.D.}$$

Note that $\bar{F}_i(t)+[f_i*\bar{G}]_{(t)}$ is the probability that the first unit of the ith component will either survive to time t or will fail in (0,t] and not be replaced by time t. If we define the

random variable $N_i(t)$ to be the total number of cycles (up time plus replacement time) of component i in (0,t], then $P[N_i(t) = 0] = \bar{F}_i(t)+[f_i*\bar{G}]_{(t)}$. In the following results we use this interpretation for Corollary 5-3.

Corollary 5-4.

Suppose all the assumptions described in Corollary 5-3 hold. Let $A_I^{(n)}(t)$ and $A_{II}^{(n)}(t)$ be the availabilities of the first and second system respectively and assume also at time 0 that both systems start to operate from up states. Then,

$$A_I^{(n)}(t) \geq A_{II}^{(n)}(t) \qquad t \geq 0 .$$

Proof of Corollary 5-4.

We have

$$A_I^{(n)}(t) = \bar{F}_1(t) + \sum_{k=1}^{n} [(f_1*g)^{(k)}*\bar{F}_1]_{(t)} \tag{5-8}$$

and

$$A_{II}^{(n)}(t) = \bar{F}_2(t) + \sum_{k=0}^{n} [(f_2*g)^{(k)}*\bar{F}_2]_{(t)} \tag{5-9}$$

Let X_i be the random variable which describes the sum of an uptime and a replacement time for component i. Since $\lambda_2 \geq \lambda_1$ and $\eta_1 = \eta_2$, we have $X_1 \overset{st}{\geq} X_2$ which is equivalent to

$$\mathrm{Prob}\{X_1 \geq t\} \geq \mathrm{Prob}\{X_2 \geq t\}$$

or

$$\mathrm{Prob}\{N_1(t) \leq k\} \geq \mathrm{Prob}\{N_2(t) \leq k\} \tag{5-10}$$

where

$$\mathrm{Prob}\{N_i(t) \leq k\} = \sum_{\ell=0}^{k} [(f_i * g)^{(\ell)} * (\bar{F}_i + f_i * \bar{G})]_{(t)} \tag{5-11}$$

Define

$$h_i^{(k)}(\cdot) = (f_i * g)^{(k)}(\cdot)$$

which is the density function for the sum of k cycles for component i. Rewriting Eq. (5-11) yields

$$P\{N_i(t) \leq k\} = \sum_{\ell=0}^{k} \int_0^t h_i^{(\ell)}(s) \cdot [P\{N_i(t-s) = 0\} \cdot \frac{\bar{F}_i(t-s)}{P\{N_i(t-s)=0\}}$$

$$+ P\{N_i(t-s) = 0\} \cdot \frac{[f_i * \bar{G}]\,(t-s)}{P\{N_i(t-s) = 0\}}]ds \tag{5-12}$$

Define

$$\frac{\bar{F}_i(t-s)}{P\{N_i(t-s) = 0\}} = \gamma_{i\,up}(t,s)$$

and

$$\frac{[f_i*\bar{G}]_{(t-s)}}{P\{N_i(t-s)=0\}} = \gamma_{i\ dn}(t,s)\ .$$

Then we have

$$\gamma_{i\ up}(t,s) + \gamma_{i\ dn}(t,s) = 1 \tag{5-13}$$

and from Corollary 5-3,

$$\gamma_{1\ up}(t,s) \geq \gamma_{2\ up}(t,s) \tag{5-14}$$

for $0 \leq s \leq t$.

The term $h_i^{(\ell)}(s) \cdot P\{N_i(t-s)=0\}$ represents the probability density function that there are exactly ℓ complete cycles in $[0,t]$ and the ℓth terminal point of a cycle occurs at s. Thus in the interval $[s,t]$ there is no complete cycle. Given this, the probability that the system will be found in state "up" at time t is $\bar{F}_i(t-s)$ and the probability that the process will be found in state "down" (under replacement) is $[f_i*\bar{G}](t-s)$. Continuing the proof of Corollary 5-4, we rewrite Eq. (5-12) as

$$P\{N_i(t) \leq n\} = \int_{s=0}^{t} [\sum_{k=0}^{n} h_i^{(k)}(s) \cdot P\{N_i(t-s)=0\}[\gamma_{i\ up}(t,s)$$

$$+ \gamma_{i\ dn}(t,s)]ds$$

$$P\{N_i(t) \le n\} = \int_0^t [\sum_{k=0}^{n} h_i^{(k)}(s)] \cdot P\{N_i(t-s) = 0\} ds \tag{5-15}$$

Then, from (5-10) and (5-15) we get

$$\int_0^t [\sum_{k=0}^{n} h_1^{(k)}(s)] \cdot P\{N_1(t-s) = 0\} ds$$

$$\ge \int_0^t [\sum_{k=0}^{n} h_2^{(k)}(s)] \cdot P\{N_2(t-s) = 0\} ds \ .$$

Multiplying both sides of this relationship by a non-negative function does not change the inequality, so we find that

$$\begin{aligned} A_{II}^{(n)}(t) &= \int_0^t [\sum_{k=0}^{n} h_2^{(k)}(s)] \cdot P\{N_2(t-s) = 0\} \cdot \gamma_{2\,up}(t,s) ds \\ &\le \int_0^t [\sum_{k=0}^{n} h_1^{(k)}(s)] \cdot P\{N_1(t-s) = 0\} \cdot \gamma_{2\,up}(t,s) ds \\ &\le \int_0^t [\sum_{k=0}^{n} h_1^{(k)}(s)] \cdot P\{N_1(t-s) = 0\} \cdot \gamma_{1\,up}(t,s) ds \\ &= A_I^{(n)}(t) \ . \end{aligned}$$

Q.E.D.

Corollary 5-5.

Let $A_{iD_p}^{(n,m)}(t)$ $(i = 1, 4, 5,\ p = 1,2)$ be the availability at time t of a two-component system under scenario i when the system starts to operate from state D_p with (n,m) spares available at time 0. Then

$$A_5^{(n,m)}(t) \le A_1^{(n,m)}(t) \Longleftrightarrow A_{5D_p}^{(n,m)}(t) \le A_{1D_p}^{(n,m)}, \text{ and}$$

$$A_1^{(n,m)}(t) \le A_4^{(n,m)}(t) \Longleftrightarrow A_{1D_p}^{(n,m)} \le A_{4D_p}^{(n,m)}(t),\ p = 1,2$$

Proof of Corollary 5-5.

For the proof of

$$A_5^{(n,m)}(t) \leq A_1^{(n,m)}(t) \Longleftrightarrow A_{5D_p}^{(n,m)}(t) \leq A_{1D_p}^{(n,m)}(t)$$

we let

$$\dot{f}_i(t) = \theta_i\lambda_i e^{-\theta_i\lambda_i t}$$

under scenario 5 (i.e., $\theta_i > 1, \quad i = 1,2$). The recursive equations for the availabilities are:

$$A_1^{(n,m)}(t) = \bar{F}_1\bar{F}_2(t) + [f_1\bar{F}_2 * A_{1D_1}^{(n,m)}](t)$$
$$+ [f_2\bar{F}_1 * A_{1D_2}^{(n,m)}](t) \tag{5-16}$$

and

$$A_5^{(n,m)}(t) = \bar{F}_1\bar{F}_2(t) + [f_1\bar{F}_2 * A_{5D_1}^{(n,m)}](t)$$
$$+ [f_2\bar{F}_1 * A_{5D_2}^{(n,m)}](t) \tag{5-17}$$

From Eq. (5-16) and Eq. (5-17) we have

$$A_1^{(n,m)}(t) - A_5^{(n,m)}(t) = [f_1\bar{F}_2 * \{A_{1D_1}^{(n,m)} - A_{5D_1}^{(n,m)}\}](t)$$
$$+ [f_2\bar{F}_1 * \{A_{1D_2}^{(n,m)} - A_{5D_2}^{(n,m)}\}](t) \tag{5-18}$$

Suppose

$$A_{1D_1}^{(n,m)}(\cdot) \geq A_{5D_1}^{(n,m)}(\cdot)$$

and

$$A_{1D_2}^{(n,m)}(\cdot) \geq A_{5D_2}^{(n,m)}(\cdot) ,$$

then from Eq. (5-18)

$$A_1^{(n,m)}(t) \geq A_5^{(n,m)}(t) .$$

Conversely suppose

$$A_1^{(n,m)}(t) \geq A_5^{(n,m)}(t) \qquad t \geq 0.$$

Then from Eq. (5-18) at least one of the two terms of the right hand side must be positive. Thus it is sufficient to show that

$$A_{1D_1}^{(n,m)}(t) \geq A_{5D_1}^{(n,m)}(t) \Longleftrightarrow A_{1D_2}^{(n,m)}(t) \geq A_{5D_2}^{(n,m)}(t) .$$

That is, if one of the terms of the right hand side of Eq. (5-18) is positive, then the other term is positive whenever

$$A_1^{(n,m)}(t) \geq A_5^{(n,m)}(t) .$$

Assume without loss of generality that

$$A_{1D_1}^{(n,m)}(t) \geq A_{5D_1}^{(n,m)}(t) \qquad t \geq 0 .$$

We know that by conditioning on the first repair time of component 2 under each scenario,

$$A_{1D_2}^{(n,m)} = \sum_{k=0}^{n-1}[\{g_2[(f_1*g_1)^{(k)}*\bar{F}_1]\}*A_1^{(n-k,m-1)}](t)$$

$$+ \sum_{k=0}^{n-1}[\{g_2[(f_1*g_1)^{(k)}*f_1*\bar{G}_1]\}*A_{1D_1}^{(n-k,m-1)}](t) \qquad (5\text{-}19)$$

and

$$A_{5D_2}^{(n,m)} = \sum_{k=0}^{n-1}[\{g_2[(\mathring{f}_1*g_1)^{(k)}*\bar{F}_1^{\circ}]\}*A_5^{(n-k,m-1)}](t)$$

$$+ \sum_{k=0}^{n-1}[\{g_2[(\mathring{f}_1*g_1)^{(k)}*\mathring{f}_1*\bar{G}_1]\}*A_{5D_1}^{(n-k,m-1)}](t) \quad . \qquad (5\text{-}20)$$

From Eq. (5-10) we have

$$\sum_{k=0}^{n-1}[(f_1*g_1)^{(k)}*\bar{F}_1](u) + \sum_{k=0}^{n-1}[(f_1*g_1)^{(k)}*f_1*\bar{G}_1](u)$$

$$\geq \sum_{k=0}^{n-1}[(\mathring{f}_1*g_1)^{(k)}*\bar{F}_1^{\circ}](u) + \sum_{k=0}^{n-1}[(\mathring{f}_1*g_1)^{(k)}*\mathring{f}_1*\bar{G}_1](u) \quad .$$

$$(5\text{-}21)$$

Multiplying both sides of Eq. (5-21) by $g_2(u)$ which assumes positive value for all values of $u \geq 0$ yields

$$\sum_{k=0}^{n-1}\{g_2[(f_1*g_1)^{(k)}*\bar{F}_1]\}(u) + \sum_{k=0}^{n-1}\{g_2[(f_1*g_1)^{(k)}*f_1*\bar{G}_1]\}(u)$$

$$\geq \sum_{k=0}^{n-1}\{g_2[(\mathring{f}_1*g_1)^{(k)}*\bar{F}_1^{\circ})\}(u) + \sum_{k=0}^{n-1}\{g_2[(\mathring{f}_1*g_1)^{(k)}*f_1*\bar{G}_1]\}(u) .$$

$$(5\text{-}22)$$

For ease of writing let

$$\{g_2[(f_1*g_1)^{(k)}*\bar{F}_1]\}_{(u)} = a_1^{(k)}(u)$$

$$\{g_2[(f_1*g_1)^{(k)}*f_1*\bar{G}_1]\}_{(u)} = a_2^{(k)}(u)$$

$$\{g_2[(\mathring{f}_1*g_1)^{(k)}*\mathring{\bar{F}}_1]\}_{(u)} = b_1^{(k)}(u)$$

$$\{g_2[(\mathring{f}_1*g_1)^{(k)}*\mathring{f}_1*\bar{G}_1]\}_{(u)} = b_2^{(k)}(u)$$

then the expression (5-22) can be written as

$$\sum_{k=0}^{n-1} a_1^{(k)}(u) + \sum_{k=0}^{n-1} a_2^{(k)}(u) \geq \sum_{k=0}^{n-1} b_1^{(k)}(u) + \sum_{k=0}^{n-1} b_2^{(k)}(u) \tag{5-23}$$

or

$$\sum_{k=0}^{n-1} a_1^{(k)}(u) - \sum_{k=0}^{n-1} b_1^{(k)}(u) \geq \sum_{k=0}^{n-1} b_2^{(k)}(u) - \sum_{k=0}^{n-1} a_2^{(k)}(u)\,. \tag{5-24}$$

By the assumption of

$$A_1^{(k,\ell)}(t) \geq A_5^{(k,\ell)}(t) \text{ for all integers } k,\ell \geq 0,\ t \geq 0.$$

and by the inequality of (5-24), convolution operations show

$$\sum_{k=0}^{n-1}[a_1^{(k)}*A_1^{(n-k,m-1)}]_{(t)} - \sum_{k=0}^{n-1}[b_1^{(k)}*A_5^{(n-k,m-1)}]_{(t)}$$

$$\geq \sum_{k=0}^{n-1}[a_1^{(k)}*A_1^{(n-k,m-1)}]_{(t)} - \sum_{k=0}^{n-1}[b_1^{(k)}*A_1^{(n-k,m-1)}]_{(t)}$$

$$\geq \sum_{k=0}^{n-1}[b_2^{(k)}*A_1^{(n-k,m-1)}]_{(t)} - \sum_{k=0}^{n-1}[a_2^{(k)}*A_1^{(n-k,m-1)}]_{(t)}$$

$$\geq \sum_{k=0}^{n-1}[b_2^{(k)}*A_{1D_1}^{(n-k,m-1)}]_{(t)} - \sum_{k=0}^{n-1}[a_2^{(k)}*A_{1D_1}^{(n-k,m-1)}]_{(t)}$$

$$\geq \sum_{k=0}^{n-1}[b_2^{(k)}*A_{5D_1}^{(n-k,m-1)}]_{(t)} - \sum_{k=0}^{n-1}[a_2^{(k)}A_{1D_1}^{(n-k,m-1)}]_{(t)} \qquad (5\text{-}25)$$

Rewriting expression (5-25) yields

$$\sum_{k=0}^{n-1}[a_1^{(k)}*A_1^{(n-k,m-1)}]_{(t)} + \sum_{k=0}^{n-1}[a_2^{(k)}*A_{1D_1}^{(n-k,m-1)}]_{(t)}$$

$$\geq \sum_{k=0}^{n-1}[b_1^{(k)}*A_5^{(n-k,m-1)}]_{(t)} + \sum_{k=0}^{n-1}[b_2^{(k)}*A_{5D_1}^{(n-k,m-1)}]_{(t)}$$

which we recognize as

$$A_{1D_2}^{(n,m)}(t) \geq A_{5D_2}^{(n,m)}$$

from the expressions (5-19) and (5-20). We have shown that whenever

$$A_1^{(n,m)}(t) \geq A_5^{(n,m)}(t)$$

holds. It follows that

$$A_{1D_1}^{(n,m)}(t) \geq A_{5D_1}^{(n,m)}$$

implies

$$A_{1D_2}^{(n,m)}(t) \geq A_{5D_2}^{(n,m)}(t)$$

if $\quad A_1^{(n,m)}(t) \geq A_5^{(n,m)}(t)$ holds.

Thus, we have shown that

$$A_1^{(n,m)}(t) \geq A_5^{(n,m)}(t) \quad \text{and}$$

$$A_{1D_p}^{(n,m)}(t) \geq A_{5D_p}^{(n,m)}(t) \quad \text{are equivalent}$$

for p = 1,2.

In a similar fashion it can be shown that

$$A_1^{(n,m)}(t) \leq A_4^{(n,m)}(t) \quad \text{and}$$

$$A_{1D_p}^{(n,m)}(t) \leq A_{4D_p}^{(n,m)}(t) \quad \text{are equivalent}$$

for p = 1,2, which completes the proof of Corollary 5-5.

Theorem 5-6.

$$A_5^{(n,m)}(t) \leq A_1^{(n,m)}(t) \leq A_4^{(n,m)}(t) \qquad t \geq 0, \quad m,m \in N$$

Proof.

For the proof of

$$A_5^{(n,m)}(t) \leq A_1^{(n,m)}(t)$$

Let

$$\overset{\circ}{f}_i(t) = \theta_i \lambda_i e^{-\theta_i \lambda_i t} \quad \text{for} \quad \theta_i > 1 \ , \quad i = 1,2.$$

and i = 1,2. When (n,m) = (0,0)

$$A_5^{(0,0)}(t) = A_1^{(0,0)}(t) = e^{-(\lambda_1+\lambda_2)t} ,$$

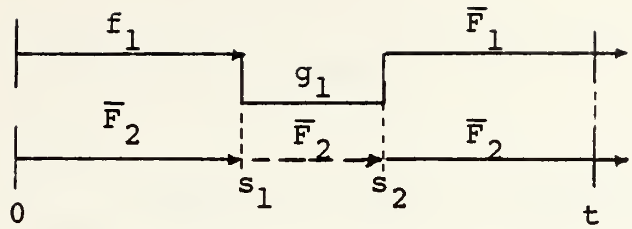
and when (n,m) = (1,0)

$$A_5^{(1,0)}(t) = \bar{F}_1\bar{F}_2(t) + [f_1\bar{F}_2 * g_1\overset{\circ}{\bar{F}}_2 * \bar{F}_1\bar{F}_2]_{(t)} \leq A_1^{(1,0)}(t)$$

$$= \bar{F}_1\bar{F}_2 + [f_1\bar{F}_2 * g_1\bar{F}_2 * \bar{F}_1\bar{F}_2]_{(t)} .$$

The last inequality holds because

$$[g_1\bar{F}_2]_{(\tau)} \geq [g_1\overset{\circ}{\bar{F}}_2]_{(\tau)} \qquad \tau \geq 0 .$$



Also, when (n,m) = (0,1) we have

$$A_5^{(0,1)}(t) \leq A_1^{(0,1)}(t) .$$

We will assume that

$$A_5^{(i,j)}(t) \leq A_1^{(i,j)}(t)$$

for (i,j) < (n,m) and show that

$$A_5^{(n,m)}(t) \leq A_1^{(n,m)}(t) .$$

From Eq. (5-5) we have

$$A_1^{(n,m)}(t) = \bar{F}_1\bar{F}_2(t) + \int_0^t \sum_{k=0}^{m-1} [f_1\bar{F}_2*\{g_1[(f_2*g_2)^{(k)}*\bar{F}_2]\}](s_1)$$

$$\cdot A_1^{(n-1,m-k)}(t-s_1)ds_1$$

$$+ \int_0^t \sum_{k=0}^{m-1} [f_1\bar{F}_2*\{g_1[(f_2*g_2)^{(k)}*f_2*\bar{G}_2]\}](s_1)$$

$$\cdot A_{1D_2}^{(n-1,m-k)}(t-s_1)ds_1$$

$$+ \int_0^t \sum_{\ell=0}^{n-1} [f_2\bar{F}_1*\{g_2[(f_1*g_1)^{(\ell)}*\bar{F}_1]\}](s_2)$$

$$\cdot A_1^{(n-\ell,m-1)}(t-s_2)ds_2$$

$$+ \int_0^t \sum_{\ell=0}^{n-1} [f_2\bar{F}_1*\{g_2[(f_1*g_1)^{(\ell)}*f_1*\bar{G}_1]\}](s_2)$$

$$\cdot A_{1D_1}^{(n-\ell,m-1)}(t-s_2)ds_2 \cdot \qquad (5\text{-}26)$$

In a similar fashion, we set up

$$
\begin{aligned}
A_5^{(n,m)}(t) = {} & \bar{F}_1\bar{F}_2 + \int_0^t \sum_{k=0}^{m-1} [f_1\bar{F}_2 * \{g_1[(\overset{\circ}{f}_2 * g_2)^{(k)} * \overset{\circ}{\bar{F}}_2]\}]_{(s_1)} \\
& \cdot A_5^{(n-1,m-k)}(t-s_1)ds_1 \\
& + \int_0^t \sum_{k=0}^{m-1} [f_1\bar{F}_2 * \{g_1[(\overset{\circ}{f}_2 * g_2)^{(k)} * \overset{\circ}{f}_2 * \bar{G}_2]\}]_{(s_1)} \\
& \cdot A_{5D_2}^{(n-1,m-k)}(t-s_1)ds_1 \\
& + \int_0^t \sum_{\ell=0}^{n-1} [f_2\bar{F}_1 * \{g_2[(\overset{\circ}{f}_2 * g_2)^{(\ell)} * \overset{\circ}{\bar{F}}_1]\}]_{(s_2)} \\
& \cdot A_5^{(n-\ell,m-1)}(t-s_2)ds_2 \\
& + \int_0^t \sum_{\ell=0}^{n-1} [f_2\bar{F}_1 * \{g_2[(\overset{\circ}{f}_2 * g_2)^{(\ell)} * f_1 * \bar{G}_1]\}]_{(s_2)} \\
& \cdot A_{5D_1}^{(n-\ell,m-1)}(t-s_2)ds_2
\end{aligned}
\qquad (5\text{-}27)
$$

For ease of writing we let

$$a_1^{(k)}(s) = [f_1\bar{F}_2 * \{g_1[(f_2 * g_2)^{(k)} * \bar{F}_2]\}]_{(s)}$$

$$b_1^{(k)}(s) = [f_1\bar{F}_2 * \{g_1[(f_2 * g_2)^{(k)} * f_2 * \bar{G}_2]\}]_{(s)}$$

$$a_2^{(\ell)}(s) = [f_2\bar{F}_1 * \{g_2[(f_1 * g_1)^{(\ell)} * \bar{F}_1]\}]_{(s)}$$

$$b_2^{(\ell)}(s) = [f_2\bar{F}_1 * \{g_2[(f_1 * g_1)^{(\ell)} * f_1 * \bar{G}_1]\}]_{(s)}$$

and we let $a_1^{\circ(k)}$, $b_1^{\circ(k)}(s)$, $a_2^{\circ(\ell)}(s)$, $b_2^{(\ell)}(s)$ be the corresponding terms in Eq. (5-27).

To prove that

$$A_1^{(n,m)}(t) \geq A_5^{(n,m)}(t)$$

we must show that

$$\sum_{k=0}^{m-1}\{[a_1^{(k)}*A_1^{(n-1,m-k)}]_{(t)} + [b_1^{(k)}*A_{1D_2}^{(n-1,m-k)}]_{(t)}\}$$
$$\geq \sum_{k=0}^{m-1}\{[a_1^{\circ(k)}*A_5^{(n-1,m-k)}]_{(t)} + [b_1^{\circ(k)}*A_{5D_2}^{(n-1,m-k)}]_{(t)}\} \tag{5-28}$$

and

$$\sum_{\ell=0}^{n-1}\{[a_2^{(\ell)}*A_1^{(n-\ell,m-1)}]_{(t)} + [b_2^{(\ell)}*A_{1D_1}^{(n-\ell,m-1)}]_{(t)}\}$$
$$\geq \sum_{\ell=0}^{n-1}\{[a_2^{\circ(\ell)}*A_5^{(n-\ell,m-1)}]_{(t)} + [b_2^{\circ(\ell)}*A_{5D_1}^{(n-\ell,m-1)}]_{(t)}\} \tag{5-29}$$

This is accomplished beginning with expression (5-10) which implies that

$$\sum_{k=0}^{m-1}[(f_2*g_2)^{(k)}*\overline{F}_2]_{(\mu)} + \sum_{k=0}^{m-1}[(f_2*g_2)^{(k)}*f_2*\overline{G}_2]_{(\mu)}$$
$$\geq \sum_{k=0}^{m-1}\{[(f_2^{\circ}*g_2)^{(k)}*\overline{F}_2^{\circ}]_{(\mu)} + [(f_2^{\circ}*g_2)^{(k)}*f_2^{\circ}*\overline{G}_2]_{(\mu)}\} \tag{5-30}$$

Multiplying this expression by $g_1(u)$ and taking convolution by $f_1F_2(\cdot)$ on both sides of (5-30) yields

$$\sum_{k=0}^{m-1}[a_1^{(k)}](s_1) + \sum_{k=0}^{m-1}[b_1^{(k)}](s_1) \geq \sum_{k=0}^{m-1}[a_1^{\circ(k)}](s_1) + \sum_{k=0}^{m-1}[b_1^{\circ(k)}](s_1) \qquad (5\text{-}31)$$

Since

$$A_1^{(i,j)}(\cdot) \geq A_5^{(i,j)}(\cdot)$$

for $(i,j) < (n,m)$, convolution operations show that

$$\begin{aligned}
&\sum_{k=0}^{m-1}\{[a_1^{(k)}*A_1^{(n-1,m-k)}](t) - [a_1^{\circ(k)}*A_5^{(n-1,m-k)}](t)\}\\
&\geq \sum_{k=0}^{m-1}\{[a_1^{(k)}*A_1^{(n-1,m-k)}](t) - [a_1^{\circ(k)}*A_1^{(n-1,m-k)}](t)\}\\
&\geq \sum_{k=0}^{m-1}\{[b_1^{\circ(k)}*A_1^{(n-1,m-k)}](t) - [b_1^{(k)}*A_1^{(n-1,m-k)}](t)\}\\
&\geq \sum_{k=0}^{m-1}\{[b_1^{\circ(k)}*A_{1D_2}^{(n-1,m-k)}](t) - [b_1^{(k)}*A_{1D_2}^{(n-1,m-k)}](t)\}\\
&\geq \sum_{k=0}^{m-1}\{[b_1^{\circ(k)}*A_{5D_2}^{(n-1,m-k)}] - [b_1^{(k)}*A_{1D_2}^{(n-1,m-k)}](t)\}.
\end{aligned}$$

The second inequality above comes from (5-31), the third inequality comes from

$$A_1^{(i,j)}(\cdot) \geq A_{1D_2}^{(i,j)}(\cdot)$$

and the last inequality comes from Corollary (5-5). Thus, we have verified (5-28). In a similar fashion we can verify the expression (5-29). Thus, we have proved

$$A_5^{(n,m)}(t) \leq A_1^{(n,m)}(t)$$

for $t \geq 0$ and $n,m \in \mathbb{N}$.

Following the exact procedures used in the proof of

$$A_5^{(n,m)}(t) \leq A_1^{(n,m)}(t)$$

it can be shown that the inequality

$$A_1^{(n,m)}(t) \leq A_4^{(n,m)}(t)$$

is true, proving Theorem 5-6.

Corollary 5-7:

$$A_{5D_p}^{(n,m)}(t) \leq A_{1D_p}^{(n,m)}(t) \leq A_{4D_p}^{(n,m)}(t) .$$

From Theorem 5-6 we have $A_5^{(n,m)}(t) \leq A_1^{(n,m)}(t)$. By Corollary 5 - 5 this implies that $A_{5D_p}^{(n,m)} \leq A_{1D_p}^{(n,m)}(t)$. Also, we have $A_1^{(n,m)}(t) \leq A_4^{(n,m)}(t)$ which implies $A_{1D_p}^{(n,m)}(t) \leq A_{4D_p}^{(n,m)}(t)$. Thus

$$A_{5D_p}^{(n,m)}(t) \leq A_{1D_p}^{(n,m)}(t) \leq A_{4D_p}^{(n,m)} .$$

VI. ALLOCATION OF SPARE PARTS

A. BACKGROUND

In the previous chapters we have defined operational availability; we have derived computational formulae for the point availability when only finitely many spare parts are available; we have looked at the impact of the operational deployment scenario and configuration on system point availability; and we have determined order relationships for system availability under different scenarios. The original motivation for this research effort was to develop a methodology for allocating scarce resources (in this problem, dollars spent for spares) so as to maximize system availability. In this chapter we develop such an algorithm which utilizes the results obtained in the earlier chapters to optimally allocate dollars for spare parts for the components within a given system. Depending on one's definition of the "system", this algorithm can be applied toward the allocation of spare parts for, say, an entire ship or for the individual components of a weapon such as a radar or a fire control system.

Various authors have considered resource allocation of spares (parallel redundancy) in order to maximize system reliability with a budget constraint. Moskowitz and McLean [30] used a variational method to optimize redundancy. Burton and Howard [7] and Bellman and Dreyfus [6] solved the problem using dynamic programming. Kettelle [22] developed a heuristic algorithm

from dynamic programming and that was extended by Proschan and Bray [33] to allow for multiple constraints. Ghare and Taylor [18] used a branch-and-bound procedure; Mizukami [29] used convex integer programming, and Tillman and Liittschwager [44] maximized reliability subject to several constraints using an integer programming formulation. Everett [14] used the generalized Lagrange multiplier method to solve the same problem discussed by Kettelle. Sharma and Venkateswaran [38] and Nakagawa and Nakashima [31] developed intuitive algorithms that provide approximate solutions. In all of those cases the system consists of components in series with parallel redundancy. Aggarwal [1], and Kuo, Hwang, and Tillman [24] develop heuristic methods for optimal system reliability for more complex systems. Other papers [39] and [40] have attempted to determine stock levels for components in order to maximize equipment operational availability , subject to a budget constraint. For their purposes operational availability for an equipment is calculated using the definition

$$A = \frac{MTBF}{MTBF + MTTR + MSRT}$$

where MSRT is the mean supply response time. Since MTBF and MTTR are unaffected by the number of spares, the models really minimize mean supply response time. The method they use is a Lagrange multiplier approach with an embedded dynamic programming technique.

In the research summarized above there have been many approaches to solving resource allocation problems using various mathematical programming techniques and considering different types of resource constraints. Our spares allocation problem could also be solved several different ways. The system availability objective function does not possess the required characteristics (linear, convex, concave) that allow the use of some of the specialized programming techniques. It is, however, separable and monotone in the decision variables (the numbers of spares) and the constraints are linear. Therefore, solution techniques like dynamic programming or heuristic methods can be applied.

None of the algorithms described above were developed for the availability allocation problem that we address. However, there are similarities that we can exploit. In the next section we present an allocation algorithm for the availability problem that extends the results obtained by Burton and Howard [7] and Kettelle [22].

B. THE ALLOCATION ALGORITHM

Consider a weapon system with k components, and let n_i be the number of spares allocated to component i for $i = 1,2,\ldots,k$. Let L_i and U_i be lower and upper bounds on the number of spares for component i and let $\underline{n} = (n_1,n_2,\ldots,n_k)$ be the vector of spares allocation for the k components. Finally, let $c(\underline{n})$ be

the total cost for the allocation $\underline{n}$ and B the upper bound on the dollars available for allocation. The mathematical problem we address is

$$\begin{aligned} &\max \quad A^{(\underline{n})}(t) \\ &\text{subject to} \quad c(\underline{n}) \leq B \\ &\qquad\qquad\qquad L_i \leq n_i \leq U_i \\ &\qquad\qquad\qquad i = 1,2,\ldots,k \end{aligned} \tag{6-1}$$

where $A^{(\underline{n})}(t)$ is the system point availability with allocation $\underline{n}$.

The mathematical programming problem (6-1) has little in the way of special structure in the objective function which allows the use of various efficient linear or nonlinear programming algorithms. However the system availability does possess a separability and monotonic structure that allows the use of dynamic programming methods. The solution technique that we derive is based on Kettelle's reliability redundancy allocation algorithm [22].

Kettelle provides an easily usable algorithm for obtaining an exact solution for maximizing the reliability of a parallel redundancy series system subject to a budget constraint. His algorithm generates undominated redundancy allocations (or dominating sequence of allocations) for successively larger subsystems from undominated allocations for small subsystems.

To understand the concept of "an undominated allocation" the following definition is introduced [5].

Definition: n° is undominated (or dominating) if

$$A^{(\underline{n})}(t) > A^{(\underline{n}^\circ)}(t) \quad \text{implies} \quad c(\underline{n}) > c(\underline{n}^\circ)$$

whereas

$$A^{(n)}(t) = A^{(\underline{n}^\circ)}(t) \quad \text{implies either} \quad c(\underline{n}) > c(\underline{n}^\circ)$$

$$\text{or} \quad c(\underline{n}) = c(\underline{n}^\circ)$$

where

$$\underline{n} = (n_1, n_2, \ldots, n_N) .$$

Kettelle confined his algorithm to series-type systems. Burton & Howard [7] showed that the allocation problem can be solved by dynamic programming for any system configuration composed of a mixture of series and parallel connections. They developed a computer algorithm for this problem and demonstrated that the dynamic programming method works very well for complex systems.

The Burton-Howard recursive dynamic programming algorithm can be improved computationally by introducing Kettelle's idea of undominated sequences and his tableau computation methods. The algorithm which we develop for optimizing system availability

adapts features of both the Kettelle algorithm and the Burton-Howard algorithm and results in an algorithm which is computationally efficient. The result solves the allocation problem not only for the budget B, but also for all budgets $B' \leq B$. This is important since in real world applications the budget B itself is not always known precisely at the time of solution. We describe the algorithm in the following material. We assume that the computational formulae derived earlier are used to evaluate component availability and that the system configuration is known. A numerical example is provided in Section VI.C.

The steps of the availability allocation algorithm are described below. Since the algorithm was generated out of a dynamic programming solution, it is not necessary to prove optimality. (Proofs are available from the Kettelle and Burton-Howard references.) Where we have made modifications of the previous results to reduce the number of decision alternatives at a given point, we prove that the modifications cannot result in inferior solutions.

The Algorithm

Define the N stages ($N \leq k$) to consist of the independent system entities. These entities can be the individual components if they operate independently or modules composed of dependently operating components (such as two components in series operating under scenario 2). Let $a_i^{(n_i)}$ be the availability (for convenience we drop reference to time t even though there is still

a specific time t in which we are interested) for the independent entity i, i = 1,2,...,N.

1) Compute the stage return, $a_i^{(n_i)}$ for stage i for $L_i \le n_i \le U_i$. This calculation utilizes the results derived in Chapters II through V.

2) Formulate the N-stage return function using the availability calculus for independent series/parallel configurations. Burton and Howard show that the problem can always be formulated so that separability and monotonicity are satisfied for any series/parallel mixed system configuration provided the entities are independent.

3) Set up a tableau such as that shown in Tables 6.7 for the two-stage problem. The entries in the row headings correspond to the triple, $(n_1, n_1c_1$, and $a_1^{(n_1)}$); the entries in the column heading correspond to $(n_2, n_2c_2, a_2^{(n_2)})$. The entries in the body of the table give the allocation (n_1,n_2), the cost $c(n_1,n_2)$ and the two-stage return (availability).

4) Start with (1,1) as the first element in the sequence of undominated allocations for the subsystem consisting of stages one and two.

5) Select as the next member of the sequence of dominating allocations the cheapest cost entry with availability higher than the previous element of the sequence. (We discuss later some methods of eliminating entire blocks of possibilities from consideration.)

6) After proceeding through a given tableau in the above manner, increase the problem to a 3-stage return using the

undominated allocations from the previous 2-stage problem as the row entries and $(n_3, c_3(n_3), a_3^{(n_3)})$ as the column entries. Obtain the dominating allocations for this subsystem consisting of stages 1, 2, and 3 as before.

7) In general, for the N-stage problem, use as the row elements $((n_1, n_2, \ldots, n_{N-1}), c(n_1, n_2, \ldots, n_{N-1}), A^{(n_1, n_2, \ldots, n_{N-1})})$ and as the column elements $(n_N, c(n_N), a^{n_N})$ and step through selecting a sequence of dominating allocations by taking the cheapest cost entry with availability higher than the previous value.

Note that if c_{ij} and a_{ij} represent the cost and availability of cell (i,j), and if cell (i,j) is a member of the optimal sequence, then the next member of the optimal sequence must be located in the region represented as quadrant I or III in Figure 6.1 below.

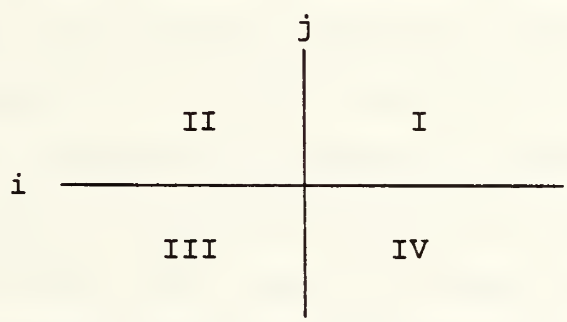


Fig. 6.1 Definition of quadrant I, II, III, IV

The tableau method used in this algorithm has the following advantages:

(1) It does not need backtracking procedures which are used in the standard dynamic programming algorithm.

(2) It does not need extra calculations for the changes of budget level. Since we can read off Figure 6.3, the allocation and availability for each level of the budget ranges from 0 to 53.

(3) It is efficient especially for manual calculations. As we could see in the previous tables (Tables 6.7 ~ 6.10) we do not need to fill out every cell of the tables. Heuristic inspection of some of the cells allows the user to eliminate large blocks of cells. As an example, suppose, in Table 6.9, we have just computed elements of the cell (9,7) of the table 6.9 which shows allocation (3,4,0,0,8), cost 17.4 and availability 0.6336, and we found out that this allocation should currently be included in the undominated sequence according to the 5th criterion of this algorithm, then we know that we have already considered all the elements of the cells above the solid line in Table 6.9, all of which have costs less than 17.4. So in the next calculations, we may try the entries in several cells in quadrant I or III for the possible elimination of entire blocks. Suppose for example we compute the availability in the cell (19,5) of Table 6.9 which we found to be 0.5966 (lower than 0.6336) with cost 24.6 (higher than 17.4), then we do not need to consider cells which are contained in the shaded area ("North-West Corner" elimination rule). This is because cost and availability of the cell (19,5) of Table 6.9 are maximal among those of the cells contained in the shaded area (cost and availability in the cell (i,j) of the table are

increasing in i and/or j), and thus every cell in the shaded area has lower availability with higher cost than those of cell (9,7) which is the current element in the dominating sequence.

In those stages where cumulative stage returns are computed from series connections, the remainder of an entire row or column can sometimes be rejected as dominated. Let c_{io} and a_{io} represent the cost and availability of the entry heading row i. If $a_{ij} > a_{i'o}$ where $i' < i$, all entires in row i' which cost more than c_{ij} are dominated because $a_{ij} > a_{i'o} > a_{i'k}$ $k \geq 1$. The second inequality follows because $a_{i'k} = a_{i'o} \cdot a_{ok}$ where $a_{ok} \leq 1$, $k = 1,2,\ldots$. The same is true for columns. As an example, note that in Table 6.7 cell (2,2) (cost 5.4, availability 0.330135) dominates every cell in the first row starting with (1,4) since $a_{22} = 0.330135 > a_{1o} = 0.2865 > a_{1k}$ $(k \geq 1)$, and $c_{22} = 5.4 < c_{1\ell}$, $\ell \geq 3$.

In those stages where cumulative stage returns are computed from parallel connections there seems to be no clear cut rule of elimination other than the "north-west corner elimination rule", i.e., try several cells heuristically in quadrant I and III, if $a_{i'j'} < a_{ij}$ then we eliminate from further consideration cell (ℓ,m) such that $\ell \leq i'$ and $m \leq j'$.

C. AN EXAMPLE

As an illustration consider the system configuration shown in Figure 6.2 and the data in Table 6.1. Solve the following problem.

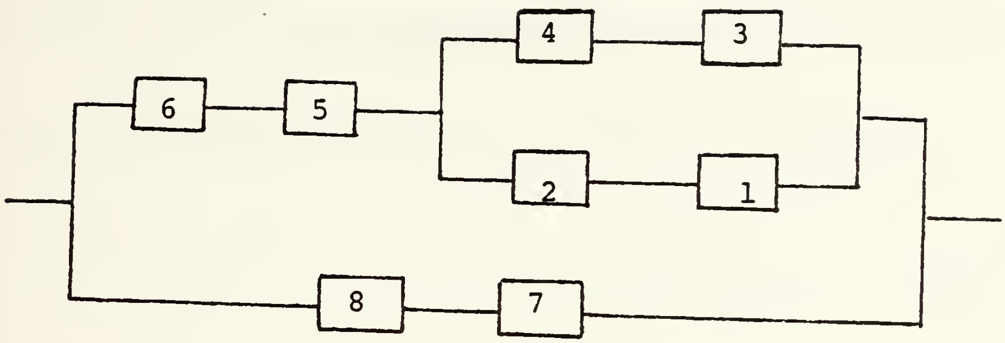


Fig. 6.2 System configuration for example problem

Table 6.1 System Data

Component		λ	η	c	u	n	a	operating rule
①. ②	①	$\frac{1}{80}$	$\frac{1}{4}$	1.4	7	n_1	a_1	scenario 1
	②	$\frac{1}{60}$	$\frac{1}{3}$	1.3	7	n_2	a_2	spares not shared
③. ④	③	$\frac{1}{50}$	$\frac{1}{2}$	1.5	4	n_3	a_{34}	scenario 2
	④	$\frac{1}{40}$	$\frac{1}{2}$	1.2	6	n_4		spares not shared
⑤. ⑥		$\frac{1}{50}$	$\frac{1}{4}$	1.0	8	n_{56}	a_{56}	scenario 2 spares shared
⑦. ⑧		$\frac{1}{40}$	$\frac{1}{2}$	1.4	9	n_{78}	a_{78}	scenario 1 spares shared

The problem is:

$$\max \quad A^{(n)}(t)$$

$$\text{s.t.} \quad n \cdot c \leq B$$

$$0 \leq n_i \leq U_i$$

for t = 100, B is flexible between 40 and 50.

Solution: For this problem the availability is given by

$$A^{(n_1,n_2,n_3,n_4,n_{56},n_{78})} = 1 - [1 - a_{78}^{(n_{78})}][1 - a_{56}^{(n_{56})}]$$

$$\cdot\{1 - (1 - a_{34}^{(n_3,n_4)})(1 - a^{(n_1)} a_2^{(n_2)})\}] \tag{6-2}$$

which is separable since it can be written as

$$A^{(n_1,n_2,n_3,n_4,n_{56},n_{78})} = a_{78} \circ a_{56} \circ a_{34} \circ a_2 \circ a_1$$

where "o" represents the composition operator.

We seek

$$A(B) = \max_n A^{(n_1,n_2,n_3,n_4,n_{56},n_{78})}$$

$$\text{s.t.} \quad n \cdot c \leq B$$

$$0 \leq n_i \leq U_i \ .$$

Let $a_i^{(n_i)}$ be the stage return and $A_i(X_i)$ be the maximum i-stage return.

For stage 1,

$$A_1(X_1) = \max_{n_1} a_1^{(n_1)}$$

$$\text{s.t.} \quad 0 \leq n_1 \leq 6 \qquad (6\text{-}3)$$

$$0 \leq c_1 n_1 \leq X_1$$

For stages 2, 3, 4 and 5

$$A_2(X_2) = \max_{n_2} a_2^{(n_2)} \cdot A_1(X_2 - n_2 c_2)$$

$$\text{s.t.} \quad 0 \leq n_2 \leq 6 \qquad (6\text{-}4)$$

$$0 \leq n_2 c_2 \leq X_2$$

$$A_{34}(X_{34}) = \max_{n_3, n_4} \{1 - [1 - a_{34}^{(n_3, n_4)}][1 - A_2(X_{34} - n_3 c_3 - n_4 c_4)]\}$$

$$\text{s.t.} \quad 0 \leq n_3 \leq 3$$

$$0 \leq n_4 \leq n_5 \qquad (6\text{-}5)$$

$$n_3 c_3 + n_4 c_4 \leq X_{34}$$

$$A_{56}(X_{56}) = \max_{n_{56}} a_{56}^{(n_{56})} \cdot A_{34}(X_{56} - c_{56} n_{56})$$

$$\text{s.t.} \quad 0 \leq n_{56} \leq 6 \qquad (6\text{-}6)$$

$$0 \leq c_{56} n_{56} \leq X_{56}$$

$$A_{78}(X_{78}) = 1 - \min_{n_{78}} [1 - a_{78}^{(n_{78})}][1 - A_{56}(X_{78} - c_{78}n_{78})]$$

$$\text{s.t.} \quad 0 \leq n_{78} \leq 7 \qquad (6\text{-}7)$$

$$0 \leq c_{78}n_{78} \leq X_{78} = B$$

The stage returns ($a_i^{(n_i)}$) are computed from the formulas in Chapter IV and the computation results are listed in Tables (6.2 - 6.6). Here n_i is the total number of parts (original plus spares for component i).

At stage 1 and 2, from Eq. (6-3) and Table 6.2 the maximum return from stage 1 is

$$A_1(n_1c_1) = a_1^{(n_1)}$$

To obtain a complete sequence of undominated allocations for the subsystem consisting of stages 1 and 2 according to recursive Equation (6-4), we set up Table 6.7. The entires in the body of the Table (6.5) give the spares, (vector of two elements), cost, and availability for the subsystem consisting of stages 1 and 2. Thus, the entry (2,3) corresponds to $n_1 = 2$ and $n_2 = 3$ with cost 6.7 achieving subsystem availability 0.495725. The chosen elements connected by arrows form an undominated sequence of allocations.

The elements of the sequence are chosen in the following way.

(1) Start with (1.1), the first undominated allocation.

(2) The next undominated allocation is the cheapest cost entry with availability higher than that of previous allocations.

Table 6.2 Stage return from stage 1

n_1	$n_1 c_1$	$a_1^{(n_1)}$
1	1.4	0.2865
2	2.8	0.6476
3	4.2	0.8565
4	5.6	0.9305
5	7.0	0.9379
6	8.4	0.9412
7	9.8	0.9417
∞	∞	0.9524

Table 6.3 Stage return from stage 2

n_2	$n_2 C_2$	$a_2^{(n_2)}$
1	1.3	0.1889
2	2.6	0.5098
3	3.9	0.7655
4	5.2	0.8929
5	6.5	0.9377
6	7.8	0.9495
7	9.1	0.9519
∞	∞	0.9524

Table 6.4 Stage return from stage 3

n_4 / $n_4 C_4$ — n_3 / $n_3 C_3$	1 1.2	2 2.4	3 3.6	4 4.8	5 6.0	6 7.2
1 1.5	(1, 1) 2.7 0.0111	(1, 2) 3.9 0.0418	(1, 3) 5.1 0.0802	(1, 4) 6.3 0.1116		
2 3.0	(2, 1) 4.2 0.0350	(2, 2) 5.4 0.1271	(2, 3) 6.6 0.2409	(2, 4) 7.8 0.3315		
3 4.5	(3, 1) 5.7 0.0595	(3, 2) 6.9 0.2120	(3, 3) 8.1 0.3968	(3, 4) 9.3 0.5407	(3, 5) 10.5 0.6213	
4 6.0				(4, 4) 10.8 0.7160	(4, 5) 12.0 0.8124	(4, 6) 13.2 0.8533

Table 6.5 Stage return from stage 4

n_{56}	$n_{56}C_{56}$	$a_{56}^{(n_{56})}$
2	2.0	0.01832
3	3.0	0.1034
4	4.0	0.2723
5	5.0	0.4854
6	6.0	0.6617
7	7.0	0.7747
8	8.0	0.8285

Table 6.6 Stage return from stage 5

n_{78}	$n_{78}C_{78}$	$a_{78}^{(n_{78})}$
2	2.8	0.0079
3	4.2	0.0426
4	5.6	0.1300
5	7.0	0.2741
6	8.4	0.4483
7	9.8	0.6133
8	11.2	0.7410
9	12.6	0.8238

Table 6.7 Sequence of max return from stage 1, 2

a_1 \ a_2	1 1.3 0.1889	2 2.6 0.5098	3 3.9 0.7655	4 5.2 0.8929	5 6.5 0.9377	6 7.8 0.9495	7 9.1 0.9519
1 1.4	(1, 1) 2.7 0.0541	(1, 2) 4.0 0.1461	(1, 3) 5.3 0.2193	6.6 0.2558			
2 2.8	4.1 0.1223	(2, 2) 5.4 0.3301	(2, 3) 6.7 0.4957	(2, 4) 8.0 0.5783	(2, 5) 9.3 0.6072		
3 4.2			(3, 3) 8.1 0.6556	(3, 4) 9.4 0.7648	(3, 5) 10.7 0.8031		
4 5.6				(4, 4) 10.8 0.8309	(4, 5) 12.1 0.8725	(4, 6) 13.4 0.8835	(4, 7) 14.7 0.8858
5 7.0						(5, 6) 14.8 0.8906	(5, 7) 16.1 0.8928
6 8.4						(6, 6) 16.2 0.8937	(6, 7) 17.5 0.8960
7 9.8	11.1 0.1779	12.4 0.4800	13.7 0.7208	15.0 0.8408	16.3 0.8830	17.6 0.8941	(7, 7) 19.9 0.8964

These procedures are consistent with the Burton-Howard algorithm. The sequence of undominated allocation in Table 6.7 shows that if we have 15 as our budget for this two-component subsystem then we allocate $n_1 = 5$ and $n_2 = 6$ with cost 14.8 achieving availability 0.89055, if B = 20 then n = (7,7) with cost 19.9 achieving 0.8964 and so forth.

At stage 3, component #3 and #4 form a two-component series subsystem which operates under scenario 2 with spares not shared. The stage returns are computed from the formula (4-22) and (4-23) and listed in Table 6.4. The sequence of undominated allocations is marked by arrows also in Table 6.4. From recursive equation (6-5) we obtain a sequence of undominated allocations for the subsystem consisting of component #1, #2, #3 and #4 by using the undominated allocations of stage 1, 2 and the undominated allocations of stage 3 which is shown in Table 6.8.

At stage 4, since component #5 and #6 operate under scenario 2 with spares shared by both components, the payoff function $a_{56}^{(n_{56})}$ whihc is computed from formula (4-27) and (4-28) and listed in Table 6.5 is the availability of the module which is an independent entity.

According to the recursive equations (6-6) we obtain a sequence of undominated allocations for the subsystem consisting of components #1, #2, #3, #4, #5, and #6 by using the undominated allocations of stages 1, 2, 3 and the sequence of payoffs from stage 4 (component #5, #6). Table 6.9 shows this

sequence , and Table 6.10 shows final allocations according to the recursive equations (6-7) in a similar fashion.

From the final table (6.10) we can construct the following graph which shows allocation and availability for each level of budget which is flexible ranging from 40 to 50.

If we would like to have a system availability of 0.900 then we would need only a budget of 26.7 consumed by allocating $(n_1, n_2, n_3, n_4, n_{56}, n_{78}) = (3,3,0,0,6,9)$. Figure 6.3 is the graphical representation of Table 6.10.

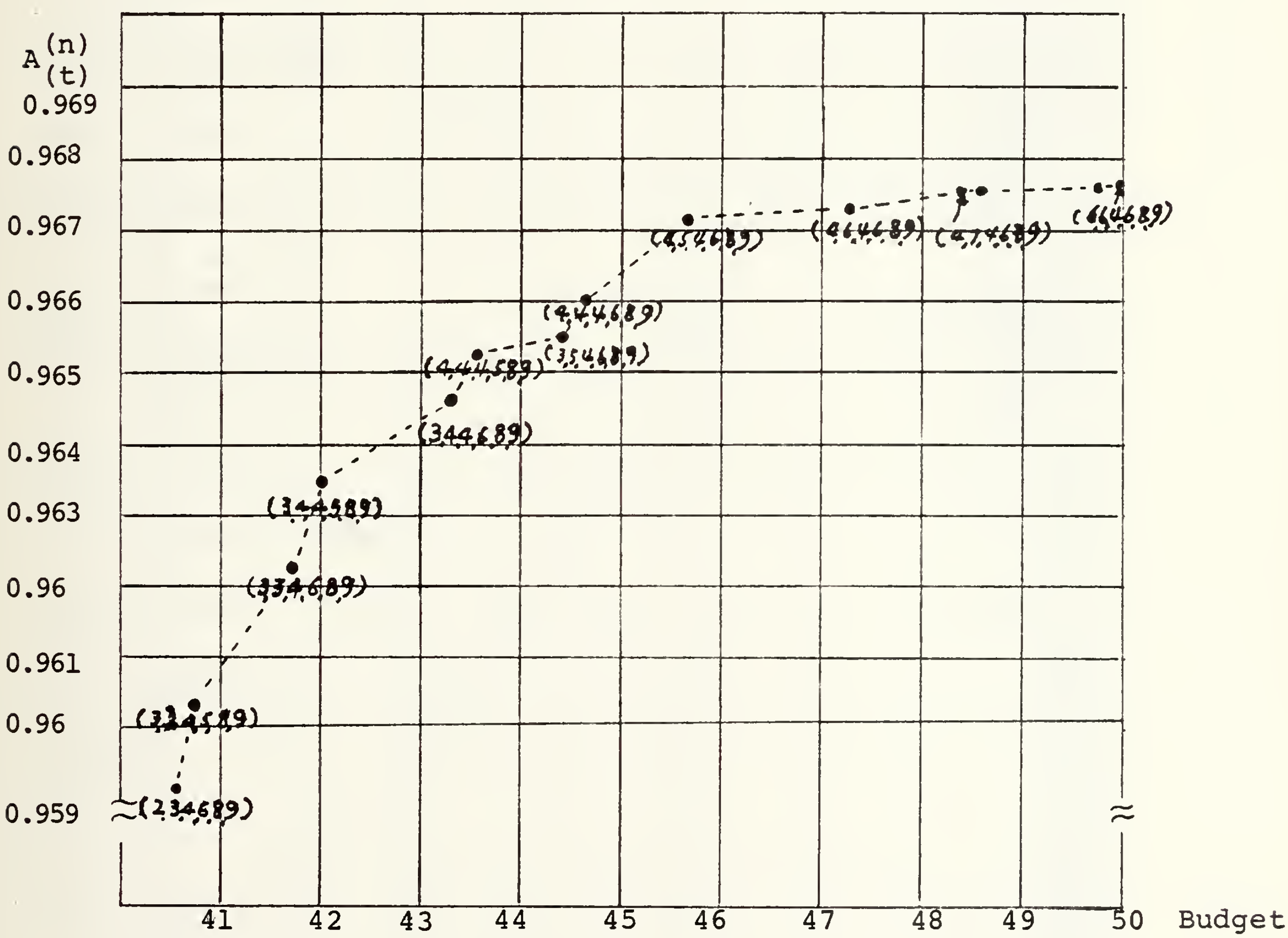


Fig. 6.3 Availability-cost curve for undominated allocations (budget level 40 - 50).

Table 6-8: Sequence of Max Return from Stage 1, 2, 3

a_{34} \ A_{12}	(0.0) 0 0.0	(1.1) 2.7 0.0111	(1.2) 3.9 0.0418	(1,3) 5.1 0.08016	(2.2) 5.4 0.1271	(2,3) 6.6 0.2409	(2.4) 7.8 0.3315	(3,3) 8.1 0.3968	(3,4) 9.3 0.5407	(3,5) 10.5 0.6213	(4.4) 10.8 0.7160	(4,5) 12.0 0.8124	(4,6) 13.2 0.8533
(0,0) 0 0.0	(0,0.0.0) 0 0.0	(0,0,1,1) 2.7 0.0111	(0,0,1,2) 3.9 0.0418	(0,0,1,3) 5.1 0.0802	(0,0,2,2) 5.4 0.1271	(0,0,2,3) 6.6 0.2409	(0,0,2,4) 7.8 0.3315	(0,0,3,3) 8.1 0.3968	(0,0,3,4) 9.3 0.5407	(0,0,3,5) 10.5 0.6213	(0,0,4,4) 10.8 0.7160	(0,0,4,5) 12.0 0.8124	(0,0,4,6) 13.2 0.8533
(1.1) 2.7 0.0541	(1,1,0.0) 2.7 0.0541	(1,1,1,1) 5.4 0.0646	(1,1,1,2) 6.6 0.0937	(1,1,1,3) 7.8 0.1299	(1,1,2,2) 8.1 0.1743	9.3 0.2826	10.5	10.8 0.429	12.0 0.5656	13.2	13.5 0.7313	14.7	15.9
(1.2) 4.0 0.1461	(1,2,0,0) 4.0 0.1461	(1,2,1,1) 6.7 0.1555	(1,2,1,2) 7.9 0.1817	9.1	9.4					14.5 0.6766	14.8		17.2 0.8747
(1.3) 5.3 0.2193	(1,3,0,0) 5.3 0.2193	8.0 0.228	9.2 0.2530						14.6				18.5 0.8854
(2, 2) 5.4 0.3301	(2,2,0.0) 5.4 0.3301	8.1 0.3376	9.3 0.358						14.7			17.4 0.8743	(2,2,4,6) 18.6 0.9017
(2,3) 6.7 0.4957	(2,3,0,0) 6.7 0.4957	9.4 0.5013	10.6 0.5168				14.5	14.8				(2,3,4,5) 18.7 0.9054	(2,3,4,6) 19.9 0.9260
(2,4) 8.0 0.5783	(2,4,0,0) 8.0 0.5783	10.7 0.5829	11.9 0.5959			14.6						(2,4,4,5) 20.0 0.9210	(2,4,4,6) 21.2 0.9381
(3, 3) 8.1 0.6556	(3,3,0,0) 8.1 0.6556	10.8 0.6594			13.5	14.7						(3,3,4,5) 20.1 0.9354	(3,3,4,6) 21.3 0.9495
(3,4) 9.4 0.7648	(3,4,0,0) 9.4 0.7648	12.1 0.7674	13.3	14.5								(3,4,4,5) 21.4 0.9559	(3,4,4,6) 22.6 0.9655
(3,5) 10.7 0.8031	(3,5,0,0) 10.7 0.8031	13.4 0.8053	14.6									22.7 0.9630	(3,5,4,6) 23.9 0.9710
(4,4) 10.8 0.8309	(4,4,0,0) 10.8 0.8309	13.5 0.8327	14.7									(4,4,4,5) 22.8 0.9683	(4,4,4,6) 24.0 0.9752
(4,5) 12.1 0.8725	(4,5,0,0) 12.1 0.8725	14.8 0.8739										(4,5,4,5) 24.1 0.9761	(4,5,4,6) 25.3 0.9813
(4.6) 13.4 0.8835	(4,6,0,0) 13.4 0.8835	16.1 0.8848	17.3 0.8884	18.5 0.8928								25.4 0.9781	(4,6,4,6) 26.6 0.9829
(4.7) 14.7 0.8858	(4,7,0.0) 14.7 0.8858	17.4 0.8870										26.7 0.9786	(4,7,4,6) 27.9 0.9832
(5.6) 14.8 0.8906	(5,6,0,0) 14.8 0.8906	17.5 0.8918											(5,6,4,6) 28.0 0.9839
(5.7) 16.1 0.8928	(5,7,0,0) 16.1 0.8928												(5,7,4,6) 29.3 0.9843
(6,6) 16.2 0.8937	(6,6,0,0) 16.2 0.8937												(6,6,4,6) 29.4 0.9844
(6. 7) 17.5 0.8960	(6,7,0,0) 17.5 0.8960												(6,7,4,6) 30.7 0.9847
(7.7) 19.9 0.8964	(7,7,0,0) 19.9 0.8964												(7,7,4,6) 32.1 0.9848

Table 6-9: Sequence of Max Return from Stage 1, 2, 3, 4

A_{1234} \ a_{56}	2 2.0 0.01832	3 3.0 0.1014	4 4.0 0.2723	5 5.0 0.4854	6 6.0 0.6617	7 7.0 0.7747	8 8.0 0.8285
(0,0,0,0) 0 0.0							
(1,1,0,0) 2.7 0.0541	(1,1,0,0,2) 4.7 0.00099	(1,1,0,0,3) 5.7 0.0055	(1,1,0,0,4) 6.7 0.0147	(11,0,0,5) 7.7 0.0263	8.7 0.0358		
(1,2,0,0) 4.0 0.1460	(1,2,0,0,2) 6.0 0.00267	(1,2,0,0,3) 7.0 0.0748	(1,2,0,0,4) 8.0 0.0398	(1,2,0,0,5) 9.0 0.0709	(1,2,0,0,6) 10.0 0.0966	11.0 0.113	
(1,3,0,0) 5.3 0.2193	7.3 0.00402	8.3 0.0222	9.3 0.0597	(1,3,0,0,5) 10.3 0.1064	11.3 0.1451		
(2,2,0,0) 5.4 0.3301			(2,2,0,0,4) 9.4 0.0899	(2,2,0,0,5) 10.4 0.1603	(2,2,0,0,6) 11.4 0.2184	(2,2,0,0,7) 12.4 0.2558	13.4 0.2735
(2,3,0,0) 6.7 0.4957				(2,3,0,0,5) 11.7 0.2410	(2,3,0,0,6) 12.7 0.328	(2,3,0,0,7) 13.7 0.3840	14.7 0.4107
(2,4,0,0) 8.0 0.5783				13.0 0.2810	14.0 0.3826	(2,4,0,0,7) 15.0 0.4480	
(3,3,0,0) 8.1 0.6556				13.1 0.3182	(3,3,0,0,6) 14.1 0.4338	(3,3,0,0,7) 15.1 0.5079	(3,3,0,0,8) 16.1 0.5432
(3,4,0,0) 9.4 0.7648					15.4 0.5060	(3,4,0,0,7) 16.4 0.5925	(3,4,0,0,8) 17.4 0.6836
(3,5,0,0) 10.7 0.8031							(3,5,0,0,8) 18.7 0.6654
(4,4,0,0) 10.8 0.8309					16.8 0.5497	(4,4,0,0,7) 17.8 0.6437	(4,4,0,0,8) 18.8 0.6884
(4,5,0,0) 12.1 0.8725				17.1 0.4235	18.1	19.1 0.6759	(4,5,0,0,8) 20.1 0.7229
(4,6,0,0) 13.4 0.8835		16.4	17.4	18.4			(4,6,0,0,8) 21.4 0.7320
(4,7,0,0) 14.7 0.8858	16.7	17.7	18.7				(4,7,0,0,8) 22.7 0.7338
(5,6,0,0) 14.8 0.8906	16.8						(5,6,0,0,8) 23.8 0.7378
(5,7,0,0) 16.1 0.8928	18.1						(5,7,0,0,8) 24.1 0.7397
(6,6,0,0) 16.2 0.8937							(6,6,0,0,8) 24.2 0.7404
(6,7,0,0) 17.5 0.8960							(6,7,0,0,8) 25.5 0.7423
(2,2,4,6) 18.6 0.9017	20.6 0.0165	21.6 0.0914	22.6 0.2455	23.6 0.4377	(2,2,4,6,6) 24.6 0.5966	25.6 0.6985	(2,2,4,6,8) 26.6 0.747

a_{56} \ A_{1234}	2 2.0 0.0183	3 3.0 0.1014	4 4.0 0.2723	5 5.0 0.4854	6 6.0 0.6617	7 7.0 0.7747	8 8.0 0.8285
(2,3,4,5) 18.7 0.9054							(2,3,4,5,8) 26.7 0.750
(2,3,4,6) 19.9 0.9260							(2,3,4,6,8) 27.9 0.7672
(3,3,4,5) 20.1 0.9354							(3,3,4,5,8) 28.1 0.7750
(3,3,4,6) 21.3 0.9495							(3,3,4,6,8) 29.3 0.7866
(3,4,4,5) 21.4 0.9559							(3,4,4,5,8) 29.4 0.7920
(3,4,4,6) 22.6 0.9635							(3,4,4,6,8) 30.6 0.7999
(4,4,4,5) 22.8 0.9683							(4,4,4,5,8) 30.8 0.8022
(3,5,4,6) 23.9 0.9710							(3,5,4,6,8) 31.9 0.8045
(4,4,4,6) 24.0 0.9752							(4,4,4,6,8) 32.0 0.8079
(4,5,4,6) 25.3 0.9813							(4,5,4,6,8) 33.3 0.813
(4,6,4,6) 26.6 0.9829							(4,6,4,6,8) 34.6 0.8143
(4,7,4,6) 27.9 0.9832							(4,7,4,6,8) 35.9 0.8146
(5,6,4,6) 28.0 0.9839							(5,6,4,6,8) 36.0 0.8152
(5,7,4,6) 29.3 0.9843							(5,7,4,6,8) 37.3 0.8165
(6,6,4,6) 29.4 0.9844							(6,6,4,6,8) 37.4 0.8166
(6,7,4,6) 30.7 0.9847							(6,7,4,6,8) 38.7 0.8158
(7,7,4,6) 32.1 0.9848	34.1 0.018	35.1 0.0998	36.1 0.268	37.1 0.478	38.1 0.6516	39.1 0.7629	(7,7,4,6,8) 40.1 0.8159

Table 6-10: Sequence of Max Return from Stage 1, 2, 3, 4, 5

a_{78} / A_{123456}	0 0.0 0.0	2 2.8 0.0079	3 4.2 0.0426	4 5.6 0.130	5 7.0 0.2741	6 8.4 0.4483	7 9.8 0.6133	8 11.2 0.7410	9 12.6 0.8238
(0,0,0,0,0) 0 0.0	(0,0,0,0,0,0) 0.0 0.000	(0,0,0,0,0,2) 2.8 0.0079	(0,0,0,0,0,3) 4.2 0.0426	(0,0,0,0,0,4) 5.6 0.1300	(0,0,0,0,0,5) 7.0 0.2741	(0,0,0,0,0,6) 8.4 0.4483	(0,0,0,0,0,7) 9.8 0.6133	(0,0,0,0,0,8) 11.2 0.7410	(0,0,0,0,0,9) 12.6 0.8238
(1,1,0,0,2) 4.7 0.0010	4.7 0.0010							15.9 0.7412	(1,1,0,0,2,9) 17.3 0.8240
(1,1,0,0,3) 5.7 0.0055									(1,1,0,0,3,9) 18.3 0.8248
(1,1,0,0,4) 6.7 0.0147									(1,1,0,0,4,9) 19.3 0.82639
(1,2,0,0,3) 7.0 0.0148									(1,2,0,0,3,9) 19.6 0.82641
(1,1,0,0,5) 7.7 0.0263									(1,1,0,0,5,9) 20.3 0.8284
(1,2,0,0,4) 8.0 0.0398									(1,2,0,0,4,9) 20.6 0.8308
(1,2,0,0,5) 9.0 0.0709									(1,2,0,0,5,9) 21.6 0.8363
(2,2,0,0,4) 9.4 0.0899									(2,2,0,0,4,9) 22.0 0.8396
(1,2,0,0,6) 10.0 0.0966									(1,2,0,0,6,9) 22.6 0.8408
(1,3,0,0,5) 10.3 0.1064									(1,3,0,0,5,9) 22.9 0.8426
(2,2,0,0,5) 10.4 0.1603								21.6 0.7825	(2,2,0,0,5,9) 23.0 0.8520
(2,2,0,0,6) 11.4 0.2184									(2,2,0,0,6,9) 24.0 0.8623
(2,3,0,0,5) 11.7 0.2410									(2,3,0,0,5,9) 24.3 0.8663
(2,2,0,0,7) 12.4 0.2558									(2,2,0,0,7,9) 25.0 0.8689
(2,3,0,0,6) 12.7 0.3280									(2,3,0,0,6,9) 25.3 0.8816
(2,3,0,0,7) 13.7 0.3840									(2,3,0,0,7,9) 26.3 0.8915

a_{78} / A_{123456}	0 0.0 0.000	2 2.8 0.0079	3 4.2 0.0426	4 5.6 0.1300	5 7.0 0.2741	6 8.4 0.4483	7 9.8 0.6133	8 11.2 0.7410	9 12.6 0.8238
(3,3,0,0,6) 14.1 0.4338									(3,3,0,0,6,9) 26.7 0.900
(2,4,0,0,7) 15.0 0.4480									(2,4,0,0,7,9) 27.6 0.9027
(3,3,0,0,7) 15.1 0.5079									(3,3,0,0,7,9) 27.7 0.9133
(3,3,0,0,8) 16.1 0.5432									(3,3,0,0,8,9) 28.7 0.9195
(3,4,0,0,7) 16.4 0.5925									(3,4,0,0,7,9) 29.0 0.9282
(3,4,0,0,8) 17.4 0.6336									(3,4,0,0,8,9) 30.0 0.9354
(4,4,0,0,7) 17.8 0.6437									(4,4,0,0,7,9) 30.4 0.9372
(3,5,0,0,8) 18.7 0.6654									(3,5,0,0,8,9) 31.3 0.9410
(4,4,0,0,8) 18.8 0.6884									(4,4,0,0,8,9) 31.4 0.9451
(4,5,0,0,8) 20.1 0.7229									(4,5,0,0,8,9) 32.7 0.9512
(4,6,0,0,8) 21.4 0.7320									(4,6,0,0,8,9) 34.0 0.9528
(4,7,0,0,8) 22.7 0.7338									(4,7,0,0,8,9) 35.3 0.9531
(5,6,0,0,8) 22.8 0.7378									(5,6,0,0,8,9) 35.4 0.9538
(5,7,0,0,8) 24.1 0.7397									(5,7,0,0,8,9) 36.7 0.9541
(6,6,0,0,8) 24.2 0.7404									(6,6,0,0,8,9) 36.8 0.9543
(6,7,0,0,8) 25.5 0.7423									(6,7,0,0,8,9) 38.1 0.9546
(2,2,4,6,8) 26.6 0.7470								37.8 0.9345	(2,2,4,6,8,9) 39.2 0.9554

a_{78} / A_{123456}	0 0.0 0.000	2 2.8 0.0079	3 4.2 0.0426	4 5.6 0.1300	5 7.0 0.2741	6 8.4 0.4483	7 9.8 0.6133	8 11.2 0.7410	9 12.6 0.8238
(2,3,4,5,8) 26.7 0.7500									(2,3,4,5,8,9) 39.3 0.9560
(2,3,4,6,8) 27.9 0.7672									(2,3,4,6,8,9) 40.5 0.9590
(3,3,4,5,8) 28.1 0.7750									(3,3,4,5,8,9) 40.7 0.9604
(3,3,4,6,8) 29.3 0.7866									(3,3,4,6,8,9) 41.9 0.9624
(3,4,4,5,8) 29.4 0.7920									(3,4,4,5,8,9) 42.0 0.9633
(3,4,4,6,8) 30.6 0.7999									(3,4,4,6,8,9) 43.2 0.9647
(4,4,4,5,8) 30.8 0.8022									(4,4,4,5,8,9) 43.4 0.9651
(3,5,4,6,8) 31.9 0.8045									(3,5,4,6,8,9) 44.5 0.9655
(4,4,4,6,8) 32.0 0.8079									(4,4,4,6,8,9) 44.6 0.9660
(4,5,4,6,8) 33.3 0.8130									(4,5,4,6,8,9) 45.9 0.9671
(4,6,4,6,8) 34.6 0.8143									(4,6,4,6,8,9) 47.2 0.9673
(4,7,4,6,8) 35.9 0.8146									(4,7,4,6,8,9) 48.5 0.9673
(5,6,4,6,8) 36.0 0.8152									(5,6,4,6,8,9) 48.6 0.9674
(5,7,4,6,8) 37.3 0.8155									(5,7,4,6,8,9) 49.9 0.9675
(6,6,4,6,8) 37.4 0.8156									(6,6,4,6,8,9) 50.0 0.9675
(6,7,4,6,8) 38.7 0.8158									(6,7,4,6,8,9) 51.3 0.9675
(7,7,4,6,8) 40.1 0.8159									(7,7,4,6,8,9) 52.7 0.9676

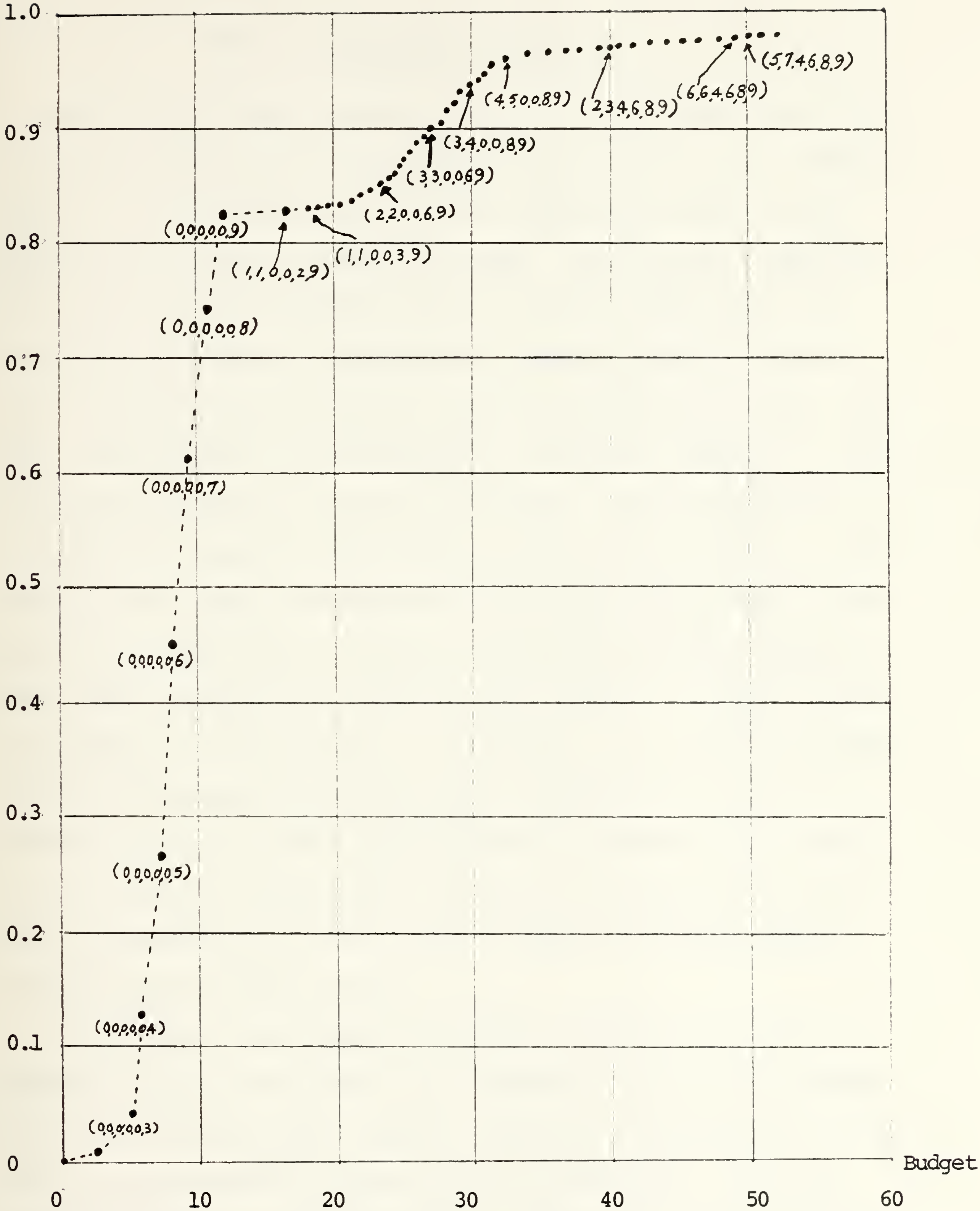


Fig. 6.4 Availability-cost curve for undominated allocations

VII. SUMMARY AND CONCLUSIONS

One of the biggest problems facing the military services is that of measuring the readiness condition of their fighting units. In this thesis we have been concerned only with the material readiness of weapon systems. The material readiness of a weapon system depends on many factors among which are equipment reliability, spare parts support, supply response time, maintenance philosophy, maintenance manning levels, system configuration, and operational deployment scenario. Many parochial measures of material readiness are calculated and reported by separate organizations within the services. For example, the supply organizations calculate indicators of the level of supply support such as fill rate or supply response times while maintenance organizations report on repair times, and others are interested in reliability. One widely reported measure of material readiness which attempts to take into account all of the avoe listed factors is operational availability. Because operational availability considers a multitude of factors, it is the type of measure on which higher command levels tend to focus.

Operational availability is widely understood to be a measure of the likelihood that a system will function successfully when called upon. However, there are differences in the way that operational availability is calculated. Military documents specify that operational availability be calculated

by taking the ratio between the mean time to failure and the sum of the mean time to failure and the mean time to replace a failed unit. This definition ignores many of the critical factors of interest and it is often not mathematically correct. Other definitions of operational availability focus on the probability that the system will be operational at a specified point in time (the point availability), the integral of point availability over a given period of time (the average availability), and the limiting probability that the system will be available (steady state availability). In this report we have concentrated on the point availability since each of the other two availability expressions can be found from point availability.

The primary objective of this research was to develop a procedure which will optimally allocate spare parts to the components of a weapon system so as to maximize the point availability of the system within a budget limit for any specified point in time. The final result of this thesis is an algorithm which accomplishes that objective.

The weapon system implied in this study is composed of components connected in series and parallel. Each component is subject to random failures and random restoration times until all replacement spares are exhausted. The failure times of each component are assumed to be independent and identically distributed with exponential distributions. Similarly, the restoration times for a component are exponential IID random variables. We have considered cases where the components

operate independently of each other and cases where the components do not operate independently. For those systems with components which all operate independently, the system availability can be determined easily after the point availability of each component has been determined as a function of the number of spares available.

For systems containing components which do not operate independently, the system point availability cannot be determined from an application of the standard reliability calculus of series/parallel/mixed systems. Instead, special consideration must be given to the type of dependence that exists among the components. Five different operating scenarios are considered for a two component series system. These operating scenarios are distinguished by what happens to the survivors while a failed component is being replaced. Under scenario one, the non-failed component continues in service, and if it fails, its replacement begins immediately and proceeds independently (the independent scenario). Under scenario two, the surviving component is shut down until replacement of the failed component is accomplished (the symmetric shut-down scenario). Under scenario three, a failure of component number one does not affect the operation of component number two, but a failure of component two will shut down component number one (the asymmetric shut-down scenario). For scenarios four and five, the failure of one component causes the failure rate of the survivor to change from λ_i to $\lambda_i\theta_i$. In scenario four $\theta_i < 1$, and in scenario five $\theta_i > 1$.

Exact expressions for the point availability of a one component system and for the marginal contribution to availability of the nth spare are determined as a function of the failure and repair parameters and the number of spares. For large values of n, normal approximation are obtained to provide simpler expressions.

The exact expression for the average availability is determined from the point availability. The tradeoff between providing a maintenance capability (implicitly equivalent to an assumption of infinitely many spares and larger replacement times) and providing modular replacement with finitely many spares is discussed. The maintenance repair rate η' that achieves the same availability provided by the replacement policy with n spares is determined. When n is small, the replacement rate η must be very large compared to the repair rate η'; however, when n gets large the ratio of η' to η approaches one.

Given a fixed length of mission duration and a finite number of spares a system may not be available at the end of a mission due to a lack of spares. The probability distribution of this downtime is determined as a function of the number of spares and mission duration. This probability distribution could be used to generate other measures of effectiveness for determining the number of spares that should be allocated to a system.

For systems containing two or more components, the system availability can be determined easily by calculations identical

to those used to calculate system reliability from component reliabilities whenever the components operate independently. All that is needed is to replace component reliabilities with component availabilities. For the cases in which the components do not operate independently and each component has only finitely many spares, the calculation of system availability is very complex. The exact expressions depend on the nature of the dependence. We explore four dependent cases in scenarios 2-5.

Exact expressions for system availability are obtained for the two-component series module described by scenarios two and three when there are infinitely many spares. It is shown that $A_1 \leq A_3 \leq A_2$ and $[Y_1] \geq E[Y_3] \geq E[Y_2]$ where A_i is the steady state availability for the case represented by scenario i and Y_i is the random variable representing system downtime. For the special case where the components in series share a common pool of finitely many spares, expressions are derived for the system availability in all five scenarios. The results obtained can be extended to series systems of more than two components with proper interpretation of the scenarios.

For the situation in which each component has its own unique pool of finitely many spares, exact computational formulae are derived for the availability of series/parallel/mixed system configuration composed of independently operating components. For scenarios two and three (symmetric shut down and asymmetric shut down) expressions are derived for the Laplace transforms

of the system point availability for any given vector (n_1,n_2) of spares allocation. Numerical examples are given for both scenarios for small values of n_1 and n_2. For larger values of n_1 and n_2 the inversion of the Laplace transforms is difficult. However, if one assumes that $\eta_1 = \eta_2$ in scenario two, an exact expression is obtained for the system availability for any vector (n_1,n_2). Other approximations for system availability are provided for scenario two whenever the repair rate is much larger than the failure rate. A numerical example of these calculations is also provided. The numerical examples point out that the system point availability is much higher when components share a common pool of spares than when the same number of spares is available but no sharing is allowed. For scenarios four and five no exact expressions are determined for system availability. However, order relationships are obtained which allow bounds to be placed on system availability.

Because some of the expressions for system availability under the different scenarios are computationally difficult it might be useful to approximate the system availability. Toward that objective, order relationships among the scenario system availabilities are obtained. The results show that

$$A_1^{(n,m)}(t) \leq A_3^{(n,m)}(t) \leq A_2^{(n,m)}(t)$$

and

$$A_5^{(n,m)}(t) \leq A_1^{(n,m)}(t) \leq A_4^{(n,m)}(t)$$

for all $t \geq 0$ and $n,m \in \mathbb{N}$ where (n,m) represents the vector of spares allocated to components 1 and 2 and the subscript refers to the scenario. The same order relationships hold when components share a common pool of spares. If the failure rates are small compared to the replacement rates, the magnitude of the differences is small. These order relationships allow $A_1^{(n,m)}(t)$ and $A_2^{(n,m)}(t)$ to be used as bounds for the exact availability for the other scenario.

The final product of this research is an algorithm for allocating resources to components of a system so as to maximize system availability for any given budget. Kettelle [22] developed an algorithm for allocating redundancy to the components of a series system where all components in series operate independently. The algorithm developed in this paper utilized Kettelle's idea of the undominated allocation sequence and his way of tableau computation to solve the more general spares allocation problem in the system availability optimization problem. The algorithm groups together those components which depend on one another as separate independent entities called modules. The expressions derived in this thesis are used to determine the availability of each module as a non-decreasing function of the resources allocated to each component of the module. The algorithm is then used to optimally allocate resources to the independent modules. The resulting

algorithm generalizes Kettelle's work by considering system availability vice reliability and by allowing for more general series/parallel/mixed system configurations. It has advantages over the standard dynamic programming algorithm by eliminating the need for backtracking and by solving the allocation problem for any budget $\leq$ B. By careful heuristic inspection the algorithm can be made very efficient for manual calculations because large blocks of cells can be eliminated from computation. In those cases where two stages are connected in serires the remainder of an entire row or column can often be eliminated from consideration. Where the cumulative stage returns are computed from parallel connections a "north-west corner" elimination rule can sometimes be applied to eliminate entire blocks of possibilities. A numerical example is provided to illustrate the allocation algorithm. The example demonstrates the elimination rules.

The exact expressionsderived in this study are sometimes computationally intractable, especially for automated solution. In actual computations some of the terms that appear in the expressions are negligible. Their omission simplifies calculation dramatically. Further work needs to be done in determining decision rules which will indicate when an expression can be ignored. Also, work should be done on the problem of numerical inversion of some of the complicated Laplace transform expressions. Another area where additional research might be fruitful is the development of a computer program to

automate the allocation algorithm. A computer program for implementing the algorithm would be easy to develop if the program is to ignore the computational advantages offered by the "elimination rules." A program which systematically searches for an opportunity to eliminate blocks of possibilities from consideration would be more efficient; albeit more difficult to develop.

This research points out that system availability depends on each of the following factors:

(1) Component reliabilities

(2) Component replacement times

(3) System configuration

(4) Operating scenario and mission profile

(5) Spare parts support and commonality of spare parts

These factors have each been considered explicitly in this thesis. A follow-on study should be made to determine what data are needed to support the allocation algorithm developed in this thesis. The study should investigate how the data should be collected. A data base management system should be developed to allow early estimation and modification of and access to the required data. An important reason for the simplistic procedures commonly used by the military services in constructing allowance lists (see, for example, the U.S. Navy's COSAL or FLSIP models [46]) is because the data requireed by efforts such as that described in this thesis are not available. Any model or algorithm can only be as good as the data

that support it. Much work needs to be done in this area before models like that developed here can be implemented operationally.

If one allows for maintenance of failed components, many additional theoretically challenging problems arise. In actual real-world situations it may be that a component is supported not only by spares, but also by a maintenance capability, where malfunctioning parts are repaired and held for reuse. The addition of a maintenance capability opens up a wealth of possibilities by allowing for different maintenance policies, number of servers, service discipline, etc. These are proposed as fruitful areas of future study.

Finally, other scenarios describing the type of dependence that may exist among components of a system should be explored.

LIST OF REFERENCES

1. Aggarwal, K.K., "Redundancy Optimization in General Systems," IEEE Trans. Rel., Vol. R-25, pp. 330-332, Dec. 1976.

2. Aggarwal, K.K., J.S. Gupta, and K.B. Misra, "A New Heuristic Criterion for Solving a Redundancy Optimization Problem," IEEE Trans. Rel., Vol. R-24, pp. 86-87, Apr. 1975.

3. Ashar, Kanu G., "Probabilistic Model of System Operation with a Varying Degree of Spares and Service Facilities," Opns. Res., Vol. 8, pp. 707-718, Sep. 1960.

4. Barlow, Richard E. and Frank Proschan, Mathematical Theory of Reliability, John Wiley and Sons, Inc., New York, 1962.

5. Barlow, Richard E. and Frank Proschan, Statistical Theory of Reliability and Life Testing, Holt, Rinehart and Winston, New York, 1975.

6. Bellman, R.E. and S.E. Dreyfus, "Dynamic Programming and Reliability of Multicomponent Devices," Opns. Res., Vol. 6, pp. 200-206, May 1958.

7. Burton, R.M. and G.T. Howard, "Optimum System Reliability for a Mixed Series and Parallel Structure," Journal of Mathematical Analysis and Applications, Vol. 28, No. 2, pp. 370-382, Nov. 1969.

8. CACI, Inc. - Federal, Optimal Operational Availability Inventory Model - Mathematical Description, Technical Report, Nov. 1978.

9. Choi, B.U., "Operational Availability with Limited Spares Support," M.S. Thesis, U.S. Naval Postgraduate School, March 1977.

10. Cinlar, Erhan, Introduction to Stochastic Processes, Prentice-Hall, Inc., New Jersey, 1975.

11. Daley, D.J., "Stochastically Monotone Markov Chains," Z. Wahrscheinlichkeitstheorie Verw, Geb. 10, pp. 305-317, 1968.

12. Derman Cyrus, and Edward Ignall, "On the Stochastic Ordering of Markov Chains," Opns. Res., Vol. 22, pp. 574-576, Nov. 1974.

13. Dreyfus, Stuart E. and Averill M. Law, The Art and Theory of Dynamic Programming, Academic Press, Inc., New York, 1977.

14. Everett H., "Generalized Lagrange Multiplier Method for Solving Problems of Optimal Allocation of Resources," Opns. Res., Vol. 11, pp. 399-417, 1963.

15. Fan, L.T., C.S. Wang, F.A. Tillman, and C.L. Hwang, "Optimization of System Reliability," IEEE Trans. Rel., Vol. R-16, pp. 81-86, Sep. 1967.

16. Federowicz, A.J. and M. Mazumdar, "Use of Geometric Programming to Maximize Reliability Achieved by Redundancy," Opns. Res., Vol. 16, pp. 948-954, Sep.-Oct. 1968.

17. Gaver, D.P., "Time to Failure and Availability of Paralleled Systems with Repair," IEEE Trans. Rel., Vol. R-12, pp. 30-38, June 1963.

18. Ghare, P.M. and R.E. Taylor, "Optimal Redundancy for Reliability in Series System," Opns. Res., Vol. 17, pp. 838-847, Sep. 1969.

19. Gnedenko, B.V., "Some Theorems on Standbys," Proc. Fifth Berkeley Symposium on Mathematical Statistics and Probability, University of California Press, Vol. 3, pp. 285-291, 1967.

20. Gopal, K., K.K. Aggarwal, and J.S. Gupta, "An Improved Algorithm for Reliability Optimization," IEEE Trans. Rel., Vol. R-27, No. 5, pp. 325-328, Dec. 1978.

21. Kalmykov, G.I., "On the Partial Ordering of One-Dimensional Markov Processes," Theor. Probab. Appl. 7, pp. 456-459, 1962.

22. Kettelle, J.D., "Least-Cost Allocation of Reliability Investment," Opns. Res., Vol. 10, pp. 249-265, May 1967.

23. Kumagai, M., "Availability of an n-spare System with a Single Repair Facility," IEEE Trans. Rel., Vol. R-24, No. 3, pp. 216-217, Aug. 1975.

24. Kuo, W., C.L. Hwang, and F.A. Tillman, "A Note on Heuristic Methods in Optimal System Reliability," IEEE Trans. Rel., Vol. R-27, No. 5, pp. 320-324, Dec. 1978.

25. Liang, T.Y., "Availability of a Special 2-unit Series System," IEEE Trans. Rel., Vol. R-27, No. 4, pp. 294-297, Oct. 1978.

26. Lie, C.H., C.L. Hwang, and F.A. Tillman, "Availability of Maintained Systems: State of the Art Survey," IEEE Trans. Rel., AIIE Trans., Vol. 9, No. 3, pp. 247-259, Sep. 1977.

27. Misra, K.B., "A Simple Approach for Constrained Redundancy Optimization Problems," IEEE Trans. Rel., Vol. R-21, pp. 30-34, Feb. 1972.

28. Misra, K.B., "A Method of Solving Redundancy Optimization Problem," IEEE Trans. Rel., Vol. R-20, pp. 117-120, Aug. 1971.

29. Mizukami, K., "Optimum Redundancy for Maximum System Reliability by the Method of Convex and Integer Programming," Opns. Res., Vol. 16, pp. 392-408, Mar.-Apr. 1968.

30. Moskowitz, F. and J.B. McLean, "Some Reliability Aspects of System Design," IRE Trans. Rel and Qual. Contr., Vol. PGRQC-8, pp. 7-35, Sep. 1956.

31. Nakagawa, Y. and K. Nakashima, "A Heuristic Method for Determining Optimal Reliability Allocation," IEEE Trans. Rel., Vol. R-26, No. 3, pp. 156-161, Aug. 1977.

32. NAVMAT Instruction P-4000, 1978.

33. Proschan, F. and T.H. Bray, "Optimal Redundancy Under Multiple Constraints," Opns. Res., Vol. 13, pp. 800-814, Sep. 1965.

34. Richards, F. Russell, "A Stochastic MOdel of a Repairable-Item Inventory System with Attrition and Random Lead Times," Opns. Res., Vol. 24, No. 1, pp. 118-130, Jan.-Feb. 1976.

35. Ross, Sheldon M., Applied Probability Models with Optimization Applications, Holden-Day, San Francisco, 1970.

36. Sasaki, M., S. Kaburaki, and S. Yanagi, "System Availability and Optimum Spare Unit," IEEE Trans. Rel., Vol. R-27, No. 3, pp. 182-188, Aug. 1977.

37. Schrady, D.A., "A Deterministic Inventory Model for Repairable Items," Naval Res. Log. Quart., Vol. 14, pp. 391-398, 1967.

38. Sharma, J. and K.V. Venkateswaran, "A Direct Method for Maximizing the System Reliability," IEEE Trans. Rel., Vol. R-20, pp. 256-259, Nov. 1971.

39. Sherbrooke, C.C., "METRIC: A Multi-Echelon Technique for Recoverable Item Control," Opns. Res., Vol. 16, pp. 122-141, 1968.

40. Srinivasan, V.S., "The Effect of Standby Redundancy in System's Failure with Repair Maintenance," Opns. Res., Vol. 14, pp. 1024-1036, Nov. 1966.

41. Subramanian, R., "Availability of a Gnedenko System," IEEE Trans. Rel., Vol. R-26, pp. 302-3-3, Oct. 1977.

42. Subramanian, R. and N. Ravichandran, "Availability of a Redundant System," IEEE Trans. Rel., Vol. R-27, No. 3, pp. 237-238, Aug. 1978.

43. Shulman, H.I., and H.L. Smith, "Operational Availability and Reliability Model," IEEE Trans. Rel., Vol. R-23, No. 5, pp. 290-204, Dec. 1974.

44. Tillman, F.A., and J.M. Liittschwager, "Integer Programming Formulation on Constrained Reliability Problems," Mgt. Sci., Vol. 13, pp. 887-899, July 1967.

45. Tillman, F.A., C.L. Hwang, and W. Kuo, "Determining Component Reliability and Redundancy for Optimal System Reliability," IEEE Trans. Rel., Vol. R-26, No. 3, pp. 162-165, Aug. 1977.

46. OPNAV INSTRUCTION 4441.12A, "Supply Support of the Operating Forces," 9 August 1973.

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